Dialectica, its realisers and Hoare logic

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Table of contents

	Source $\to$ Target		
Gödel ('58)	$A \in \mathrm{HA}$	$\longmapsto$	$A_D\{w, c\} \in \mathbf{T}$
		such that	
	$\vdash_{\mathrm{HA}} A$	$\Longrightarrow$	$\vdash_{\mathbf{T}} A_D\{\mathtt{M}, c\}$ *for some* $\mathtt{M} \in \mathbf{T}$

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De Paiva ('91) + Pédrot ('15)	$A \in \Lambda$	$\longmapsto$	$W(A),\ C(A) \in \mathbf{P}$
	$\mathtt{M} \in \Lambda$	$\longmapsto$	$\mathtt{M}^\bullet,\ \mathtt{M}_\mathtt{x} \in \mathbf{P}$ *(for* $\boldsymbol{x}$ *variable)*
		such that	
	$\mathtt{x} : A \vdash_\Lambda \mathtt{M} : B$	$\Longrightarrow$	$\begin{cases} \mathtt{x} : W(A) \vdash_{\mathbf{P}} \mathtt{M}^\bullet : W(B) \\ \mathtt{x} : W(A) \vdash_{\mathbf{P}} \mathtt{M}_\mathtt{x} : C(B) \to \mathcal{M}[C(A)] \end{cases}$

$A \in \Lambda \longmapsto W(A),\, C(A) \in \mathbf{P}$

	α	$E \to F$
W	α_W	$W(E) \to W(F)$ $\times$ $W(E) \times C(F) \to \mathcal{M}[C(E)]$
C	α_C	$W(E) \times C(F)$

$\mathtt{M} \in \Lambda \longmapsto \mathtt{M}^\bullet,\, \mathtt{M}_\mathtt{y} \in \mathbf{P}$

	$\mathtt{x}$	$\lambda\mathtt{x}.\mathtt{M}$	$\mathtt{PQ}$
$(\_)^\bullet$	$\mathtt{x}$	$\left\langle \begin{array}{c} \lambda\mathtt{x}.\mathtt{M}^\bullet \\ \lambda\pi.(\lambda\mathtt{x}.\mathtt{M}_\mathtt{x})\pi^1\pi^2 \end{array} \right\rangle$	$\mathtt{P}^{\bullet 1}\mathtt{Q}^\bullet$
$(\_)_\mathtt{y}$	$\begin{cases} \lambda\pi.[\pi], & \mathtt{x} = \mathtt{y} \\ \lambda\pi.0, & \mathtt{y} \neq \mathtt{y} \end{cases}$	$\lambda\pi.(\lambda\mathtt{x}.\mathtt{M}_\mathtt{y})\pi^1\pi^2$	$\lambda\pi.\left(\begin{array}{c} \mathtt{P}_\mathtt{y}\langle \mathtt{Q}^\bullet, \pi\rangle \\ + \\ \mathtt{P}^{\bullet 2}\langle \mathtt{Q}^\bullet, \pi\rangle \gg\!= \mathtt{Q}_y \end{array} \right)$

High-order Weak-Extensional Heyting-Arithmetic (WE-HA$^\omega$)

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- Axioms:

$$
\begin{gathered}
\textit{equality} \\
+ \\
PA \\
+ \\
(\mathtt{if}\ b\ \mathtt{then}\ s\ \mathtt{else}\ t = s) \vee_b (\mathtt{if}\ b\ \mathtt{then}\ s\ \mathtt{else}\ t = t) \\
+ \\
(\mathtt{rec}\ z\ y\ n = y) \vee_n (\mathtt{rec}\ z\ y\ n = z\,(n-1)\,(\mathtt{rec}\ z\ y\ (n-1)))
\end{gathered}
$$

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- Rules:

$$\begin{gathered} \textit{Intuitionistic Logic} \\ + \\ \dfrac{A_0 \to t = s \qquad A_0 \ \textit{quantifier free}}{A_0 \to B\{x := t\} \to B\{x := s\}} \end{gathered}$$

Dialectica for WE-HA$^\omega$ in WE-HA$^\omega$

$$
\begin{array}{ccccc}
\mathit{Formulas} & \longrightarrow & \mathit{q.f.Formulas} \times \overrightarrow{\mathrm{Var}} \times \overrightarrow{\mathrm{Var}} & & \\
A & \longmapsto & (\,|A|\ ,\ W(A)\ ,\ C(B)\,), & & \textit{written } |A|^{W(A)}_{C(A)}
\end{array}
$$

defined by:

$$
\begin{array}{rcll}
|A|^{\emptyset}_{\emptyset} & := & A & \textit{if } A \textit{ is atomic} \\
|A \wedge B|^{x,u}_{y,v} & := & |A|^x_y \wedge |B|^u_v & \\
|A \vee B|^{b^{\mathtt{nat}},x,u}_{y,v} & := & |A|^x_y \vee_{b^{\mathtt{nat}}} |B|^u_v & \\
|A \to B|^{f,F}_{x,v} & := & |A|^x_{Fxv} \to |B|^{fx}_v & \\
|\forall x.A|^f_{z,y} & := & |A\{x := z\}|^{fz}_y & \\
|\exists x.A|^{z,u}_y & := & |A\{x := z\}|^u_y &
\end{array}
$$

Dialectica for $\mathbf{WE\text{-}HA}^\omega$ in $\mathbf{WE\text{-}HA}^\omega$

$$\begin{array}{ccc} \textit{Formulas} & \longrightarrow & \textit{q.f.Formulas} \times \vec{\mathrm{Var}} \times \vec{\mathrm{Var}} \\ A & \longmapsto & (\,|A|\;,\,W(A)\,,\,C(B)\,), \qquad \textit{written } |A|^{W(A)}_{C(A)} \end{array}$$

Theorem (Soundness of Dialectica)

$$\textit{WE-HA}^\omega \vdash A \;\Rightarrow\; \textit{WE-HA}^\omega \vdash \forall y.\,|A|^a_y$$

where $a \in \boldsymbol{T}$ is "extracted" from the proof of A

Dialectica for WE-HA$^\omega$ in WE-HA$^\omega$

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Theorem (Soundness of Dialectica)

If WE-HA$^\omega_\Delta \supseteq$ WE-HA$^\omega$ proves the Dialectica of Δ, then:

$$\Delta + \textit{WE-HA}^\omega \vdash A \;\Rightarrow\; \textit{WE-HA}^\omega_\Delta \vdash \forall y.\, |A|^a_y$$

where $a \in \boldsymbol{T}$ is "extracted" from the proof of A

Dialectica for WE-HA$^\omega$ in WE-HA$^\omega$

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Theorem (Soundness of Dialectica)

If WE-HA$^\omega_\Delta \supseteq$ WE-HA$^\omega$ proves the Dialectica of Δ, then:

$$\Delta + \textit{WE-HA}^\omega \vdash M : A \;\Rightarrow\; \textit{WE-HA}^\omega_\Delta \vdash \forall y.\, |A|^{M^\bullet}_y$$

where $(\_) \longmapsto (\_)^\bullet$ is a program transformation like the first slides

Dialectica is a realisability interpretation

(with a stronger condition on the implication)

Theorem (Adequacy of Dialectica realisability)

If $d \Vdash \Delta$, then:

$$\Delta \vdash M : A \Rightarrow M^{\bullet}\{d\} \Vdash A$$

where $(\_) \longmapsto (\_)^{\bullet}$ is a program transformation like the first slides

Hoare Triple: $A\langle f\rangle B$

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$$\forall^{\mathrm{State}} s.(\, A \to B\{s := fs\}\,)$$

Theorem (Hoare Logic Soundness)

If the judgment $A\langle f\rangle B$ is derivable, then the formula above is provable (in some ambient theory, say WE-HA$^\omega$). So, second intuition: $f \Vdash_{Hoare} A \to B$.

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Say A and B are quantifier-free. Then the above formula is:

$$\forall^{\mathrm{State}} s.\, |\exists x.A \to \exists x.B|^{f,\emptyset}_{(s,\emptyset),\emptyset}$$

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Say A and B are quantifier-free. Then the above formula is:

$$\forall^{\text{State}} s.\, |\exists x.A \to \exists x.B|^{f,\emptyset}_{(s,\emptyset),\emptyset}$$

Let's take this seriously in all its generality:

$$A\,\langle f \mid F\rangle\, B := \forall s\, v.\, |A \to B|^{f,F}_{s,v}$$

for A, B any formula. Intuition: $\langle f \mid F\rangle \Vdash_{Dialectica} A \to B$.

Dialectica Hoare Logic (DHL)

Rules for deriving judgments $A\,\langle f \mid F\rangle\,B$, with $A, B \in \text{WE-HA}^\omega$ and $f, F \in \mathbf{T}$, such that

Theorem (Dialectica Hoare Logic Soundness)

If the judgment

$$A\,\langle f \mid F\rangle\,B$$

is derivable in DHL, then

$$\textit{WE-HA}^\omega \quad \vdash \quad \forall s\, v.\ |A|^{s}_{F s v} \to |B|^{f s}_{v}\,.$$

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If the judgment

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$$\textit{WE-HA}^\omega \quad \vdash \quad \forall s\, v.\ |A|^s_{Fsv} \to |B|^{fs}_v .$$

Usual Soundness Theorem by Gödel. But with the focus on programs f, F and DHL as a specification system for them, instead of on formulas.

See also De Paiva's thesis and Pédrot's thesis!

DHL rules

$$\bot\ \langle a \mid - \rangle\ P \qquad P\ \langle - \mid \alpha \rangle\ \top \qquad P\ \langle \mathtt{I} \mid \mathtt{proj}_2 \rangle\ P \qquad \frac{P_\exists \to Q_\forall \in \mathrm{Ax}}{P_\exists\ \langle - \mid - \rangle\ Q_\forall} \qquad \frac{P_\exists\ \langle - \mid - \rangle\ Q_\forall}{P'_\exists\ \langle - \mid - \rangle\ Q'_\forall}\ \text{for}\ \frac{P_\exists \to Q_\forall}{P'_\exists \to Q'_\forall} \in \mathrm{Rule}$$

$$\frac{P\ \langle a, b \mid \alpha \rangle\ Q \wedge R}{P\ \langle b, a \mid \tilde{\alpha} \rangle\ R \wedge Q}\, p\wedge_R \qquad \frac{P \wedge Q\ \langle a \mid \alpha, \beta \rangle\ R}{Q \wedge P\ \langle \tilde{a} \mid \tilde{\beta}, \tilde{\alpha} \rangle\ R}\, p\wedge_L \qquad \frac{P\ \langle a, b \mid \alpha \rangle\ Q \vee_c R}{P\ \langle b, a \mid \tilde{\alpha} \rangle\ R \vee_{\bar{c}} Q}\, p\vee_R \qquad \frac{P \vee_c Q\ \langle a \mid \alpha, \beta \rangle\ R}{Q \vee_{\bar{c}} P\ \langle \tilde{a} \mid \tilde{\beta}, \tilde{\alpha} \rangle\ R}\, p\vee_L$$

$$\frac{P\ \langle a \mid \alpha \rangle\ Q}{P\ \langle a, b \mid \alpha_\pi \rangle\ Q \vee_0 R}\, \vee_R \qquad \frac{P\ \langle a \mid \alpha \rangle\ Q}{P \wedge R\ \langle a_\pi \mid \alpha_\pi, \beta \rangle\ Q}\, \wedge_L \qquad \frac{P\ \langle a, b \mid \alpha \rangle\ Q \wedge R}{P\ \langle a \mid \alpha_p \rangle\ Q}\, \wedge_R \qquad \frac{P \vee_0 R\ \langle a \mid \alpha, \beta \rangle\ Q}{P\ \langle a_p \mid \alpha_p \rangle\ Q}\, \vee_L$$

$$\frac{P \wedge \phi\ \langle a \mid \alpha \rangle\ R \qquad Q \wedge \neg\phi\ \langle b \mid \beta \rangle\ R \qquad \phi\ qf}{P \vee Q\ \langle \lambda x, y.\,\mathrm{if}\ \phi\ \mathrm{then}\ ax\ \mathrm{else}\ by \mid \alpha_\pi, \beta_\pi \rangle\ R}\, cond_L \qquad \frac{P\ \langle a \mid \alpha \rangle\ Q \qquad P\ \langle b \mid \beta \rangle\ R}{P\ \langle a, b \mid \lambda x, v, w.\,\mathrm{if}\ |P|^x_{\alpha x v}\ \mathrm{then}\ \beta x w\ \mathrm{else}\ \alpha x v \rangle\ Q \wedge R}\, cond_R$$

$$\frac{P\ \langle a, b \mid \alpha \rangle\ Q \to R}{P \wedge Q\ \langle a \mid \alpha, b \rangle\ R}\, uncurry \qquad \frac{P \wedge Q\ \langle a \mid \alpha, \beta \rangle\ R}{P\ \langle a, \beta \mid \alpha \rangle\ Q \to R}\, curry \qquad \frac{P\ \langle a \mid \alpha \rangle\ Q \qquad Q\ \langle b \mid \beta \rangle\ R}{P\ \langle \lambda x.b(a(x)) \mid \lambda x, w.\ \alpha x(\beta(ax)w) \rangle\ R}\, comp$$

$$\frac{P\ \langle a \mid \alpha \rangle\ Q(t)}{P\ \langle \lambda\_.t, a \mid \alpha \rangle\ \exists x\, Q(x)}\, \exists_R \qquad \frac{P(t)\ \langle a \mid \alpha \rangle\ Q}{\forall x\, P(x)\ \langle \lambda f.a(ft) \mid \lambda\_.t, \lambda f.\alpha(ft) \rangle\ Q}\, \forall_L$$

$$\frac{P(x)\ \langle a \mid \alpha \rangle\ Q}{\exists x\, P(x)\ \langle \lambda x.a \mid \lambda x.\alpha \rangle\ Q}\, \exists_L (x \notin Q) \qquad \frac{P\ \langle a \mid \alpha \rangle\ Q(x)}{P\ \langle \lambda y, x.ay \mid \lambda y, x.\alpha y \rangle\ \forall x\, Q(x)}\, \forall_R (x \notin P)$$

$$\frac{\exists x\, P(x)\ \langle a \mid \alpha \rangle\ Q}{P(t)\ \langle at \mid \alpha t \rangle\ Q}\, s_L \qquad \frac{P\ \langle a \mid \alpha \rangle\ \forall x\, Q(x)}{P\ \langle \lambda y.ayt \mid \lambda y, v.\alpha y t v \rangle\ Q(t)}\, s_R \qquad \frac{P_\forall\ \langle a, b \mid \alpha \rangle\ \exists x\, Q(x)}{P_\forall\ \langle b \mid \alpha \rangle\ Q(a)}\, \epsilon_R \qquad \frac{\forall x\, P_\forall(x)\ \langle - \mid \alpha, \beta \rangle\ Q_{qf}}{P_\forall(\alpha)\ \langle - \mid \beta \rangle\ Q_{qf}}\, \epsilon_L$$

$$\frac{P'\ \langle \mathtt{I} \mid \mathtt{proj}_2 \rangle\ P \qquad P\ \langle a \mid \alpha \rangle\ Q \qquad Q\ \langle \mathtt{I} \mid \mathtt{proj}_2 \rangle\ Q'}{P'\ \langle a \mid \alpha \rangle\ Q'}\, cons \qquad \frac{P\ \langle a \mid \alpha \rangle\ Q \qquad a, \alpha = b, \beta}{P\ \langle b \mid \beta \rangle\ Q}\, ext \qquad \frac{P(x)\ \langle a(x) \mid \alpha(x) \rangle\ P(x+1)}{P(0)\ \langle \mathrm{rec}\ a \mid \mathrm{rec}^*\ a\alpha \rangle\ \forall x.\ P(x)}\, ind$$

Update WE-HA$^\omega$

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- Term PL: $\cdots \mid \prec : X \to X \to \mathtt{nat}$
  $\mid \mathtt{whilerec}_{\phi,a} : (X \to U) \to (X \to U \to U) \to X \to U$
- Formulas: same as before
- Axioms: same as before + the following for $\phi\{x\}$ q.f.:

$$
\begin{array}{l}
(\phi\{x := y\} \to ay \prec y) \to \\
\mathtt{whilerec}_{\phi,a}\, u\, F\, y =_U \mathtt{if}\ \phi\{x := y\}\ \mathtt{then}\ F\, y\, (\mathtt{whilerec}_{\phi,a}\, u\, F\, (ay))\ \mathtt{else}\ (uy)
\end{array}
$$

- Rules: same as before + $\dfrac{\forall x. \,(\,(\forall y \prec x. A\{x := y\}) \to A\,)}{\forall x. A}$

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Remark

The sugars

$$\begin{array}{rclccl} \mathtt{while}\ \phi\ \mathtt{do}\ a & := & \mathtt{whilerec}_{\phi,a} & \mathtt{I} & \mathtt{proj}_2 & : X \to X \\ \mathtt{while}^*\ \phi\ \mathtt{do}\ (a, \alpha) & := & \mathtt{whilerec}_{\phi,a} & \mathtt{proj}_2 & (\lambda x, f, v.\, \alpha x(fv)) & : X \to V \to V \end{array}$$

behave in WE-HA$^\omega$ like a usual *well-founded* while and a backward while, resp.

Dialectica with While

Add to DHL the rule:

$$\frac{\exists x\,(P_\forall(x) \wedge \phi(x))\,\langle a \,|\, \alpha\rangle\,\exists x\,P_\forall(x) \qquad \forall x\,(\phi(x) \to ax \prec x)}{\exists x\,P_\forall(x)\,\langle \texttt{while}\,\phi\,\texttt{do}\;a \,|\, \texttt{while}^*\,\phi\,\texttt{do}\,(a,\alpha)\rangle\,\exists x\,(P_\forall(x) \wedge \neg\phi(x))}$$

Theorem

Dialectica Hoare Logic Soundness keeps holding.

$$\exists x.\, \theta \vdash \exists x.\, (\theta \wedge \forall y \prec x. \neg\, \theta(y))$$

with $\prec$ well-founded and $\theta\{x^X\}$ quantifier-free.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\theta \wedge \phi_g\, \langle - \,|\, - \rangle\, \theta(gx)}}{\theta \wedge \phi_g\, \langle gx \,|\, - \rangle\, \exists y.\, \theta(y)}\ \exists_R}{\exists x.\, (\theta \wedge \phi_g)\, \langle g \,|\, - \rangle\, \exists y.\, \theta(y)}\ \exists_L \quad \forall x.\, (\phi_g \to gx \prec x)}{\exists x.\, \theta\, \langle \texttt{while}\ \phi_g\ \texttt{do}\ g \,|\, - \rangle\, \exists y.\, (\theta(y) \wedge \neg \phi_g)}\ \mathit{while}}{\exists x.\, \theta\, \langle \lambda x, g.(\texttt{while}\ \phi_g\ \texttt{do}\ g)x \,|\, - \rangle\, \forall g \exists y.\, (\theta(y) \wedge \neg \phi_g(y))}\ \forall_R}{\exists x.\, \theta\, \langle \lambda x, g.(\texttt{while}\ \phi_g\ \texttt{do}\ g)x \,|\, - \rangle\, \neg\neg\exists y.\, (\theta(y) \wedge \forall z \prec y.\, \neg\theta(z))}\ N}{\neg\exists y.\, (\theta(y) \wedge \forall z \prec y.\, \neg\theta(z))\, \langle - \,|\, \lambda x, g.(\texttt{while}\ \phi_g\ \texttt{do}\ g)x \rangle\, \neg\exists x.\, \theta}\ \mathit{contrapositive}$$

with $\phi_g := gx \prec x \wedge \theta(gx)$.

Idea: trial-and-error. (Appears very often in proof mining).

Fix fresh sets of commands $\overrightarrow{Comm}$, $\overleftarrow{Comm}$ and consider $\mathrm{LOOP}_D :=$ IMP with commands from above and *without* variable allocation:

$$C ::= \texttt{skip} \mid \langle c \,|\, \gamma \rangle \mid C; C \mid \texttt{if}\ \phi\ \texttt{then}\ C\ \texttt{else}\ C \mid \texttt{while}\ \phi\ \texttt{do}\ C$$

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Fix $\mathbf{T}$-types S, T and translations $\overrightarrow{Comm} \to \mathbf{T}^{S \to S}$ and $\overleftarrow{Comm} \to \mathbf{T}^{S \to T \to T}$. Define a translation $\mathrm{LOOP}_D \to \mathbf{T}^{S \to S} \times \mathbf{T}^{S \to T \to T}$:

LOOP_D	$(\_)^+$	$(\_)^-$
$\mathtt{skip}$	$\mathtt{I}$	$\mathtt{proj}_2$
$\langle c \,\|\, \gamma \rangle$	c	γ
$C_1; C_2$	$\lambda x.\, C_2^+(C_1^+ x)$	$\lambda x, w.\ C_1^- x(C_2^-(C_1^+ x)w)$
$\mathtt{if}\ \phi\ \mathtt{then}\ C_1\ \mathtt{else}\ C_2$	$\lambda s.\mathtt{if}\ \phi(s)\ \mathtt{then}\ C_1^+ s\ \mathtt{else}\ C_2^+ s$	$\lambda s, t.\mathtt{if}\ \phi(s)\ \mathtt{then}\ C_1^- st\ \mathtt{else}\ C_2^- st$
$\mathtt{while}\ \phi\ \mathtt{do}\ C$	$\mathtt{while}\ \phi\ \mathtt{do}\ C^+$	$(\mathtt{while}^*\ \phi\ \mathtt{do}\ C^+), C^-$

Hoare Logic for LOOP$_D$

$$\frac{}{[P]\,\texttt{skip}\,[P]} \qquad \frac{P(s,\gamma st) \to Q(cs,t) \in \mathrm{Ax}}{[P]\,\langle c \mid \gamma\rangle\,[Q]} \qquad \frac{[P]\,C_1\,[Q] \quad [Q]\,C_2\,[R]}{[P]\,C_1; C_2\,[R]}$$

$$\frac{[P \wedge \phi]\,C_1\,[R] \quad [Q \wedge \neg\phi]\,C_2\,[R]}{[P \vee_\phi Q]\,\textbf{if}\;\phi\;\textbf{then}\;C_1\;\textbf{else}\;C_2\,[R]} \qquad \frac{[P \wedge \phi]\,C\,[P] \quad \phi(s) \to C^+ s \prec s}{[P]\,\texttt{while}\;\phi\;\texttt{do}\;C\,[P \wedge \neg\phi]}$$

$$\frac{P'(s,t) \to P(s,t) \quad [P]\,C\,[Q] \quad Q(s,t) \to Q'(s,t)}{[P']\,C\,[Q']}$$

where the formulas and their provability are wrt the ambient WE-HA$^\omega$.

Theorem (Soundness wrt Dialectica)

Let P, Q quantifier free with only one variable s^S and one t^T. Then

$$[P]\,C\,[Q] \quad \Rightarrow \quad \begin{array}{c} \exists s \forall t. P \;\; \langle C^+ \mid C^- \rangle \;\; \exists s \forall t. Q \\ \textit{and} \\ \textit{WE-HA}^\omega \vdash \forall s, v.\ P\{t := C^- st\} \to Q\{s := C^+ s\} \end{array}$$

Big-step Operational semantics of LOOP$_D$

Forward OS: $\vec{\Downarrow} \ \subseteq (\mathbf{T}^S)^* \times \mathbf{LOOP}_D \times \mathbf{T}^S \times (\mathbf{T}^S)^* \times (\mathbf{T}^{S\to T\to T})^*$

$$\frac{}{s, \mathtt{skip} \mathrel{\vec{\Downarrow}} s, \epsilon, \epsilon} \qquad \frac{}{s, \langle c \,|\, \gamma \rangle \mathrel{\vec{\Downarrow}} cs, s :: \epsilon, \gamma :: \epsilon} \qquad \frac{s, C_1 \mathrel{\vec{\Downarrow}} s', \sigma, \Gamma \quad s', C_2 \mathrel{\vec{\Downarrow}} s'', \sigma', \Gamma'}{s, C_1 ; C_2 \mathrel{\vec{\Downarrow}} s'', \sigma' :: \sigma, \Gamma' :: \Gamma}$$

$$\frac{\phi(s) \quad s, C_1 \mathrel{\vec{\Downarrow}} s', \sigma, \Gamma}{s, \mathtt{if}\ \phi\ \mathtt{then}\ C_1\ \mathtt{else}\ C_2 \mathrel{\vec{\Downarrow}} s', \sigma, \Gamma} \qquad \frac{\neg\phi(s) \quad s, C_2 \mathrel{\vec{\Downarrow}} s', \sigma, \Gamma}{s, \mathtt{if}\ \phi\ \mathtt{then}\ C_1\ \mathtt{else}\ C_2 \mathrel{\vec{\Downarrow}} s', \sigma, \Gamma}$$

$$\frac{\neg\phi(s)}{s, \mathtt{while}\ \phi\ \mathtt{do}\ C \mathrel{\vec{\Downarrow}} s, \epsilon, \epsilon} \qquad \frac{\phi(s) \quad s, C \mathrel{\vec{\Downarrow}} s', \sigma, \Gamma \quad s' \prec s \quad s', \mathtt{while}\ \phi\ \mathtt{do}\ C \mathrel{\vec{\Downarrow}} s'', \sigma', \Gamma'}{s, \mathtt{while}\ \phi\ \mathtt{do}\ C \mathrel{\vec{\Downarrow}} s'', \sigma' :: \sigma, \Gamma' :: \Gamma}$$

Backward OS: $\overleftarrow{\Downarrow} \ \subseteq (\mathbf{T}^S)^* \times (\mathbf{T}^{S\to T\to T})^* \times \mathbf{T}^T \times (\mathbf{T}^S)^* \times (\mathbf{T}^{S\to T\to T})^* \times \mathbf{T}^T$

$$\frac{}{\sigma, \Gamma, t \mathrel{\overleftarrow{\Downarrow}} \sigma, \Gamma, t} \qquad \frac{}{s :: \sigma, \gamma :: \Gamma, t \mathrel{\overleftarrow{\Downarrow}} \sigma, \Gamma, \gamma s t} \qquad \frac{\sigma, \Gamma, t \mathrel{\overleftarrow{\Downarrow}} \sigma', \Gamma', t' \quad \sigma', \Gamma', t' \mathrel{\overleftarrow{\Downarrow}} \sigma'', \Gamma'', t''}{\sigma, \Gamma, t \mathrel{\overleftarrow{\Downarrow}} \sigma'', \Gamma'', t''}$$

Big-step Operational semantics of LOOP_D

Forward OS: $s, C \stackrel{\rightarrow}{\Downarrow} s', \sigma, \Gamma$

Backward OS: $\sigma, \Gamma, t \stackrel{\leftarrow}{\Downarrow} \sigma', \Gamma', t'$

Theorem (Forward+Backward OS = Backpropagation in LOOP_D)

Suppose that WE-HA$^\omega \vdash \forall s(\phi(s) \to C^+ s \prec s)$ for all `while` ϕ `do` C *of $LOOP_D$. Then for any $s : S$ there exist $\sigma : S^*$ and $\Gamma : (S \to T \to T)^*$ such that*

1. $$s, C \stackrel{\rightarrow}{\Downarrow} (C^+ s), \sigma, \Gamma$$
2. *for any $t : T$, $\rho : S^*$ and $\Delta : (S \to T \to T)^*$,*
$$\sigma :: \rho, \Gamma :: \Delta, t \stackrel{\leftarrow}{\Downarrow} \rho, \Delta, (C^- st).$$

Dialectica implements Automatic Differentiation: discovered by Kerjean and Pédrot!

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  Here, a, α and b, β operate in parallel on disjoint variables. So frame rule!

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Thank you, Merci, Grazie!