Computational interpretations of logic: the case of Dialectica and Realisability

Davide Barbarossa

davide.barbarossa@ens-lyon.fr

https://davidebarbarossa12.github.io/

Laboratoire de l'Informatique du Parallélisme

ENS DE LYON

Séminaire au LACL

13/04/2026

Table of contents

Let π be a formal proof of $\vdash A$

Let π be a formal proof of $\vdash A$

A proof gives evidence of the conclusion:

$$[\![\pi]\!] \Vdash A$$

Let π be a formal proof of $\vdash A$

A proof gives evidence of the conclusion:

$$[\![\pi]\!] \Vdash A$$

	Tarski	Curry-Howard	Kreisel	Krivine	Gödel
extraction language	$\{\square\}$	Type-Theory	$\mathbf{T} + \cdots$	$\lambda + \texttt{callcc} + \cdots$	$\mathbf{T} + \cdots$
$\mathtt{M} \Vdash A$	$\vDash A$	$\mathtt{M}$ *normal* $\vdash \mathtt{M} : A$	$\vdash \mathtt{M} : W(A)$ $\vdash \mathtt{M}\ \mathrm{mr}\ A$	$\mathtt{M}$ closed and p.l. $\forall \rho \in C(A),\ \mathtt{M} \perp \rho$	$\vdash \mathtt{M} : W(A)$ $\vdash \forall \rho^{C(A)}.\ \mathtt{M} \perp_A \rho$
$[\![\pi]\!] =$	$\square$	$\mathrm{nf}(\widetilde{\pi})$	$\widetilde{\pi}$	$\widetilde{\pi}$	π^+
theorem	*sound.*	*sub. red.* *+SN+confl*	*sound.*	*adequacy*	*adequacy*
$\exists$ *proof* vs $\exists$ *evidence*	$\Longleftrightarrow$	$\Longleftrightarrow$	$\not\Leftarrow, \Rightarrow$	$\not\Leftarrow, \Rightarrow$	$\not\Leftarrow, \Rightarrow$

Let π be a formal proof of $\vdash A$

A proof gives evidence of the conclusion:

$$[\![\pi]\!] \Vdash A$$

	Tarski	Curry-Howard	Kreisel	Krivine	Gödel
extraction language	$\{\square\}$	Type-Theory	$\mathbf{T} + \cdots$	$\lambda + \texttt{callcc} + \cdots$	$\mathbf{T} + \cdots$
$\texttt{M} \Vdash A$	$\vDash A$	$\texttt{M}$ *normal* $\vdash \texttt{M} : A$	$\vdash \texttt{M} : W(A)$ $\vdash \texttt{M}\ \mathrm{mr}\ A$	$\texttt{M}$ closed and p.l. $\forall \rho \in C(A),\ \texttt{M} \perp \rho$	$\vdash \texttt{M} : W(A)$ $\vdash \forall \rho^{C(A)}.\ \texttt{M} \perp_A \rho$
$[\![\pi]\!] =$	$\square$	$\mathrm{nf}(\widetilde{\pi})$	$\widetilde{\pi}$	$\widetilde{\pi}$	π^+
theorem	*sound.*	*sub. red. +SN+confl*	*sound.*	*adequacy*	*adequacy*
$\exists$ *proof* vs $\exists$ *evidence*	$\Longleftrightarrow$	$\Longleftrightarrow$	$\not\Leftarrow, \Rightarrow$	$\not\Leftarrow, \Rightarrow$	$\not\Leftarrow, \Rightarrow$

Let $[\![A]\!] := \{\texttt{M} \mid \texttt{M} \Vdash A\}$.

Question: *For a given logic, what is the theory* $\{A \mid [\![A]\!] \neq \emptyset\}$ *(and its models)?*

Let π be a formal proof of $\mathtt{x} : B \vdash A$

	Tarski	Curry-Howard	Kreisel	Krivine	Gödel
extraction language	$\{\square\}$	Type-Theory	$\mathbf{T} + \cdots$	$\lambda + \mathtt{callcc} + \cdots$	$\mathbf{T} + \cdots$
$\mathtt{M} \Vdash A$	$\models A$	$\mathtt{M}$ *normal* $\vdash \mathtt{M} : A$	$\vdash \mathtt{M} : W(A)$ $\vdash \mathtt{M}\ \mathrm{mr}\ A$	$\mathtt{M}$ closed and p.l. $\forall \rho \in C(A),\ \mathtt{M} \perp \rho$	$\vdash \mathtt{M} : W(A)$ $\vdash \forall \rho^{C(A)}.\ \mathtt{M} \perp_A \rho$
$[\![\pi]\!] =$	$\square$	$\mathrm{nf}(\widetilde{\pi})$	$\widetilde{\pi}$	$\widetilde{\pi}$	π^+
theorem	*sound.*	*sub. red. +SN+confl*	*sound.*	*adequacy*	*adequacy*
$\exists$ *proof* vs $\exists$ *evidence*	$\Longleftrightarrow$	$\Longleftrightarrow$	$\not\Leftarrow, \Rightarrow$	$\not\Leftarrow, \Rightarrow$	$\not\Leftarrow, \Rightarrow$

Let π be a formal proof of $\mathtt{x} : B \vdash A$

A proof is an algorithm transforming evidence of the hypotheses into evidence of the conclusion:

$$[\![\pi]\!] : [\![B]\!] \to [\![A]\!]$$

	Tarski	Curry-Howard	Kreisel	Krivine	Gödel
extraction language	$\{\square\}$	Type-Theory	$\mathbf{T} + \cdots$	$\lambda + \mathtt{callcc} + \cdots$	$\mathbf{T} + \cdots$
$\mathtt{M} \Vdash A$	$\vDash A$	$\mathtt{M}$ *normal* $\vdash \mathtt{M} : A$	$\vdash \mathtt{M} : W(A)$ $\vdash \mathtt{M}\ \mathrm{mr}\ A$	$\mathtt{M}$ closed and p.l. $\forall \rho \in C(A),\ \mathtt{M} \perp \rho$	$\vdash \mathtt{M} : W(A)$ $\vdash \forall \rho^{C(A)}.\ \mathtt{M} \perp_A \rho$
$[\![\pi]\!] =$	$\square$	$\mathrm{nf}(\widetilde{\pi})$	$\widetilde{\pi}$	$\widetilde{\pi}$	π^+
theorem	*sound.*	*sub. red. +SN+confl*	*sound.*	*adequacy*	*adequacy*
$\exists$ *proof* vs $\exists$ *evidence*	$\Longleftrightarrow$	$\Longleftrightarrow$	$\not\Leftarrow, \Rightarrow$	$\not\Leftarrow, \Rightarrow$	$\not\Leftarrow, \Rightarrow$

Let π be a formal proof of $\mathtt{x} : B \vdash A$

A proof is an algorithm transforming evidence of the hypotheses into evidence of the conclusion:

$$[\![\pi]\!] : [\![B]\!] \to [\![A]\!]$$

	Tarski	Curry-Howard	Kreisel	Krivine	Gödel
extraction language	$\{\square\}$	Type-Theory	$\mathrm{T} + \cdots$	$\lambda + \mathtt{callcc} + \cdots$	$\mathrm{T} + \cdots$
$\mathtt{M} \Vdash A$	$\vDash A$	$\mathtt{M}$ *normal* $\vdash \mathtt{M} : A$	$\vdash \mathtt{M} : W(A)$ $\vdash \mathtt{M}\ \mathrm{mr}\ A$	$\mathtt{M}$ closed and p.l. $\forall \rho \in C(A),\ \mathtt{M} \perp \rho$	$\vdash \mathtt{M} : W(A)$ $\vdash \forall \rho^{C(A)}.\ \mathtt{M} \perp_A \rho$
$[\![\pi]\!] =$	$\square \mapsto \square$	$\mathtt{M} \mapsto \mathrm{nf}((\lambda \mathtt{x}.\widetilde{\pi})\mathtt{M})$	$\mathtt{M} \mapsto \widetilde{\pi}\{x := \mathtt{M}\}$	$\mathtt{M} \mapsto \widetilde{\pi}\{x := \mathtt{M}\}$	$\mathtt{M} \mapsto \pi^{+}\{x := \mathtt{M}\}$
theorem	*sound.*	*sub. red. +SN+confl*	*sound.*	*adequacy*	*adequacy*
$\exists$ *proof* vs $\exists$ *evidence*	$\Longleftrightarrow$	$\Longleftrightarrow$	$\not\Leftarrow, \Rightarrow$	$\not\Leftarrow, \Rightarrow$	$\not\Leftarrow, \Rightarrow$

Realisability semantics of formulas

Fix: sets $\mathbb{W}, \mathbb{C}$, a relation $\perp \subseteq \mathbb{W} \times \mathbb{C}$, an interpretation $\mathfrak{M}$ of second order var.'s as $\mathbb{N}^k \to \mathcal{P}(\mathbb{C})$. Define functions $\mathcal{W}$: Formulas $\to \mathcal{P}(\mathbb{W})$, $\mathcal{C}$: Formulas $\to \mathcal{P}(\mathbb{C})$:

$$
\begin{aligned}
\mathcal{W}(\_) &:= \{t \in \mathbb{W} \mid t \perp \pi \ \textit{for all}\ \pi \in \mathcal{C}(\_)\} \\
\mathcal{C}(X(e_1, \ldots, e_k)) &:= X^{\mathfrak{M}}({e_1}^{\mathbb{N}}, \ldots, {e_k}^{\mathbb{N}}) \\
\mathcal{C}(A \to B) &:= \mathcal{W}(A) * \mathcal{C}(B) \\
\mathcal{C}(\forall c.A) &:= \bigcup_{m \in \mathbb{N}} \mathcal{C}(A\{c := m\}) \\
\mathcal{C}(\forall^m Y.A) &:= \bigcup_{Q : \mathbb{N}^m \to \mathcal{P}(\mathbb{C})} \mathcal{C}(A\{Y := Q\})
\end{aligned}
$$

Realisability semantics of formulas

Fix: sets $\mathbb{W}, \mathbb{C}$, a relation $\perp \subseteq \mathbb{W} \times \mathbb{C}$, an interpretation $\mathfrak{M}$ of second order var.'s as $\mathbb{N}^k \to \mathcal{P}(\mathbb{C})$. Define functions $\mathcal{W}$: Formulas $\to \mathcal{P}(\mathbb{W})$, $\mathcal{C}$: Formulas $\to \mathcal{P}(\mathbb{C})$:

$$
\begin{aligned}
\mathcal{W}(\_) &:= \{t \in \mathbb{W} \mid t \perp \pi \textit{ for all } \pi \in \mathcal{C}(\_)\} \\
\mathcal{C}(X(e_1, \ldots, e_k)) &:= X^{\mathfrak{M}}({e_1}^{\mathbb{N}}, \ldots, {e_k}^{\mathbb{N}}) \\
\mathcal{C}(A \to B) &:= \mathcal{W}(A) * \mathcal{C}(B) \\
\mathcal{C}(\forall c.A) &:= \bigcup_{m \in \mathbb{N}} \mathcal{C}(A\{c := m\}) \\
\mathcal{C}(\forall^m Y.A) &:= \bigcup_{Q : \mathbb{N}^m \to \mathcal{P}(\mathbb{C})} \mathcal{C}(A\{Y := Q\})
\end{aligned}
$$

Write $\mathtt{M} \Vdash_{\perp}^{\mathfrak{M}} A$ whenever $\mathtt{M}$ is a closed proof like term in $\mathcal{W}(A)$. Then:

	$\mathbb{W}$	$\mathbb{C}$	$\perp \subseteq \mathbb{W} \times \mathbb{C}$	$*$	$\Vdash_{\perp}^{\mathfrak{M}}$
Tarski	$\{\square\}$	$\{\dagger\}$	$\emptyset$	$\Rightarrow$	true in $\mathfrak{M}$
Krivine	Λ	Stacks(Λ)	*pole* $\perp\!\!\!\perp$	*cons*	realisable in $\mathfrak{M}$ via $\perp\!\!\!\perp$

Let Δ be our favourite theory of PA2 (or ZF/+CH/+C+...) and A our favourite formula.

Adequacy Theorem

Fix a pole and an interpretation of the 1st and 2nd order variables of Δ, A. Let $x : \Delta \vdash \mathtt{M} : A$. Then:

$$\mathtt{d} \Vdash^{\mathfrak{M}}_{\perp\!\!\!\perp} \Delta \quad \Longrightarrow \quad \mathtt{M}\{\mathtt{x} := \mathtt{d}\} \Vdash^{\mathfrak{M}}_{\perp\!\!\!\perp} A.$$

Corollary

$$\vdash \mathtt{M} : A \Longrightarrow \mathtt{M} \Vdash A$$

– *If I add the logical principles in Δ, which computational principles should I add in the programming language in order to realise the former?*
– *Which is the theory Δ of formulas that are realisable (in a given programming language)?*

Dialectica in a Nutshell

Gödel ('38/'58)	$A \in \mathrm{HA}$	$\longmapsto$	$w \perp_A c$
		such that	
	$\vdash_{\mathrm{HA}} A$	$\Longrightarrow$	*there is* $\mathtt{M} \in \mathbf{T}$ *s.t.* $\vdash_{\mathbf{T}} \forall c.\ \mathtt{M} \perp_A c$

Gödel ('38/'58)	$A \in \mathrm{HA} \quad \longmapsto \quad w \perp_A c$ *such that* $\vdash_{\mathrm{HA}} A \quad \Longrightarrow \quad$ *there is* $\mathtt{M} \in \mathbf{T}$ *s.t.* $\vdash_{\mathbf{T}} \forall c.\ \mathtt{M} \perp_A c$
Kreisel (∼'50) Kohlenbach(∼90) ...	*Several variants and extensions* (⇒ *"Proof Mining"*)

Gödel ('38/'58)	$A \in \mathrm{HA} \longmapsto w \perp_A c$ *such that* $\vdash_{\mathrm{HA}} A \Longrightarrow$ *there is* $\mathtt{M} \in \mathbf{T}$ *s.t.* $\vdash_{\mathbf{T}} \forall c.\ \mathtt{M} \perp_A c$
Kreisel (∼'50) Kohlenbach(∼90) ...	*Several variants and extensions* ($\Rightarrow$ *"Proof Mining"*)
De Paiva ('91)	*Dialectica Categories (and models of Linear Logic)*

Gödel ('38/'58)	$A \in \mathrm{HA} \longmapsto w \perp_A c$ *such that* $\vdash_{\mathrm{HA}} A \Longrightarrow$ *there is* $\mathtt{M} \in \mathbf{T}$ *s.t.* $\vdash_{\mathbf{T}} \forall c.\ \mathtt{M} \perp_A c$
Kreisel (∼'50) Kohlenbach(∼90) …	*Several variants and extensions (⇒ "Proof Mining")*
De Paiva ('91)	*Dialectica Categories (and models of Linear Logic)*
Pédrot ('14)	$A \in \mathit{source} \longmapsto W(A), C(A) \in \mathit{target}$ $\mathtt{M} \in \mathit{source} \longmapsto \mathtt{M}^\bullet, \mathtt{M}_\mathtt{x} \in \mathit{target}$ *such that* $\mathtt{x} : B \vdash_{\mathit{source}} \mathtt{M} : A \Longrightarrow \begin{cases} \mathtt{x} : W(B) \vdash_{\mathit{target}} \mathtt{M}^\bullet : W(A) \\ \mathtt{x} : W(B) \vdash_{\mathit{target}} \mathtt{M}_\mathtt{x} : C(A) \to C(B) \end{cases}$

High-order Weak-Extensional Heyting-Arithmetic (WE-HA$^\omega$)

High-order Weak-Extensional Heyting-Arithmetic (WE-HA$^\omega$)

- PL of terms: Simply typed System **T** with ground type `nat`

High-order Weak-Extensional Heyting-Arithmetic (WE-HA$^\omega$)

- PL of terms: Simply typed System **T** with ground type `nat`
- Formulas: $\top, \bot, =_{\texttt{nat}}, \wedge, \vee, \to, \forall^X, \exists^X$ and $(t =_{X \to Y} s) := \forall^X x.(tx =_Y sx)$

$$(A \vee_{b^{\texttt{nat}}} B) := ((b =_{\texttt{nat}} 0) \to A) \wedge ((b \neq_{\texttt{nat}} 0) \to B)$$

High-order Weak-Extensional Heyting-Arithmetic (WE-HA$^\omega$)

- PL of terms: Simply typed System **T** with ground type `nat`
- Formulas: $\top, \bot, =_{\texttt{nat}}, \wedge, \vee, \rightarrow, \forall^X, \exists^X$ and $(t =_{X \rightarrow Y} s) := \forall^X x.(tx =_Y sx)$

$$(A \vee_{b^{\texttt{nat}}} B) := ((b =_{\texttt{nat}} 0) \rightarrow A) \wedge ((b \neq_{\texttt{nat}} 0) \rightarrow B)$$

- Axioms:

$$\textit{equality}$$
$$+$$
$$\textit{arithmetic}$$
$$+$$
$$(\texttt{if}\ b\ \texttt{then}\ s\ \texttt{else}\ t = s) \vee_b (\texttt{if}\ b\ \texttt{then}\ s\ \texttt{else}\ t = t)$$
$$+$$
$$(\texttt{rec}\ f\ g\ n = g) \vee_n (\texttt{rec}\ f\ g\ n = f\ (n-1)\ (\texttt{rec}\ f\ g\ (n-1)))$$

High-order Weak-Extensional Heyting-Arithmetic (WE-HA$^\omega$)

- PL of terms: Simply typed System **T** with ground type `nat`
- Formulas: $\top, \bot, =_{\texttt{nat}}, \wedge, \vee, \rightarrow, \forall^X, \exists^X$ and $(t =_{X \to Y} s) := \forall^X x.(tx =_Y sx)$

$$(A \vee_{b^{\texttt{nat}}} B) := ((b =_{\texttt{nat}} 0) \rightarrow A) \wedge ((b \neq_{\texttt{nat}} 0) \rightarrow B)$$

- Axioms:

$$\begin{gathered} \textit{equality} \\ + \\ \textit{arithmetic} \\ + \\ (\texttt{if}\ b\ \texttt{then}\ s\ \texttt{else}\ t = s)\ \vee_b\ (\texttt{if}\ b\ \texttt{then}\ s\ \texttt{else}\ t = t) \\ + \\ (\texttt{rec}\ f\ g\ n = g) \vee_n (\texttt{rec}\ f\ g\ n = f\ (n-1)\ (\texttt{rec}\ f\ g\ (n-1))) \end{gathered}$$

- Rules:

$$\begin{gathered} \textit{Intuitionistic Logic} \\ + \\ \frac{A_0 \rightarrow t = s \qquad A_0\ \textit{quantifier free}}{A_0 \rightarrow B\{x := t\} \rightarrow B\{x := s\}} \end{gathered}$$

Dialectica for WE-HA$^\omega$ in WE-HA$^\omega$

$$\begin{array}{ccc} \textit{Formulas} & \longrightarrow & \textit{quant.-free Formulas} \times \mathbf{T} \times \mathbf{T} \\ A & \longmapsto & (\, w \perp_A c \;\;,\;\; W(A)\;,\; C(A)\,) \end{array}$$

Dialectica for WE-HA$^\omega$ in WE-HA$^\omega$

$$\begin{array}{ccc} \textit{Formulas} & \longrightarrow & \textit{quant.-free Formulas} \times \mathbf{T} \times \mathbf{T} \\ A & \longmapsto & (\, w \perp_A c \;,\; W(A) \,,\, C(A)\,) \end{array}$$

(Here "$\vdash$" means "provable in WE-HA$^\omega$")

Theorem (Soundness)

Let $\vdash A$. *Then*

$$\vdash \forall c.\ \mathtt{a} \perp_A c$$

where $\mathtt{a} \in \boldsymbol{T}$ *is "extracted" from the proof of* A.

Dialectica for WE-HA$^\omega$ in WE-HA$^\omega$

$$\begin{array}{ccc} \textit{Formulas} & \longrightarrow & \textit{quant.-free Formulas} \times \mathbf{T} \times \mathbf{T} \\ A & \longmapsto & (\, w \perp_A c \;\;,\;\; W(A)\;,\; C(A)\,) \end{array}$$

(Here "$\vdash$" means "provable in WE-HA$^\omega$")

Theorem (Soundness)

Let $\vdash A$. Then

$$\mathsf{a} \Vdash^{\textit{WE-HA}^\omega} A$$

where $\mathsf{a} \in \boldsymbol{T}$ is "extracted" from the proof of A.

Dialectica for WE-HA$^\omega$ in WE-HA$^\omega$

$$\begin{array}{ccc} \textit{Formulas} & \longrightarrow & \textit{quant.-free Formulas} \times \mathbf{T} \times \mathbf{T} \\ A & \longmapsto & (\, w \perp_A c \;,\; W(A) \,,\, C(A)\,) \end{array}$$

(Here "$\vdash$" means "provable in WE-HA$^\omega$")

Theorem ("Metatheorem")

Let $\Delta \vdash A$. *Then*

$$\Delta' \vdash \forall c.\ \mathtt{d} \perp_\Delta c \Longrightarrow \Delta' \vdash \forall c.\ \mathtt{a} \perp_A c$$

where $\mathtt{a} \in \boldsymbol{T}$ *is "extracted" from a proof of* A.

Dialectica for $\mathbf{WE\text{-}HA}^\omega$ in $\mathbf{WE\text{-}HA}^\omega$

$$\begin{array}{ccc} \textit{Formulas} & \longrightarrow & \textit{quant.-free Formulas} \times \mathbf{T} \times \mathbf{T} \\ A & \longmapsto & (\, w \perp_A c \;\;,\;\; W(A)\;,\; C(A)\,) \end{array}$$

(Here "$\vdash$" means "provable in WE-HA$^\omega$")

Theorem ("Metatheorem")

Let $\Delta \vdash A$. Then

$$\mathtt{d} \Vdash^{\textit{WE-HA}^\omega}_{\Delta'} \Delta \implies \mathtt{a} \Vdash^{\textit{WE-HA}^\omega}_{\Delta'} A$$

where $\mathtt{a} \in \boldsymbol{T}$ is "extracted" from a proof of A.

Dialectica for WE-HA$^\omega$ in WE-HA$^\omega$

$$\begin{array}{ccc} \textit{Formulas} & \longrightarrow & \textit{quant.-free Formulas} \times \mathbf{T} \times \mathbf{T} \\ A & \longmapsto & (\, w \perp_A c \;\; , \;\; W(A) \; , \; C(A) \,) \end{array}$$

(Here "$\vdash$" means "provable in WE-HA$^\omega$")

Theorem (Adequacy)

"Gödel" inductively defines a ***program transformation*** $(\_) \longmapsto (\_)^+$ *which meets the following specification:*

If π *proves* $\mathtt{x} : \Delta \vdash A$, *then:*

$$\mathtt{d} \Vdash^{\textit{WE-HA}^\omega}_{\Delta'} \Delta \implies \pi^+\{\mathtt{x} := \mathtt{d}\} \Vdash^{\textit{WE-HA}^\omega}_{\Delta'} A$$

Dialectica for WE-HA$^\omega$ in WE-HA$^\omega$

$$\begin{array}{ccc} \textit{Formulas} & \longrightarrow & \textit{quant.-free Formulas} \times \mathbf{T} \times \mathbf{T} \\ A & \longmapsto & (\, w \perp_A c \;,\; W(A) \;,\; C(A)\,) \end{array}$$

(Here "$\vdash$" means "provable in WE-HA$^\omega$")

Theorem (Adequacy)

"Gödel" inductively defines a ***program transformation*** $(\_) \longmapsto (\_)^+$ *which meets the following specification:*

If π proves $\mathtt{x} : \Delta \vdash A$, then:

$$\mathtt{d} \Vdash^{\textit{WE-HA}^\omega}_{\Delta'} \Delta \implies \pi^+\{\mathtt{x} := \mathtt{d}\} \Vdash^{\textit{WE-HA}^\omega}_{\Delta'} A$$

– *If I add the logical principles in Δ, which computational principles should I add in the programming language and specify by Δ', in order to realise the former?*
– *Which is the theory Δ of formulas that are interpretable (in the empty Δ' and given a programming language)?*

Dialectica for $\mathbf{WE\text{-}HA}^\omega$ in $\mathbf{WE\text{-}HA}^\omega$

$$\begin{array}{ccc} \mathit{Formulas} & \longrightarrow & \mathit{quant.\text{-}free\ Formulas} \times \mathbf{T} \times \mathbf{T} \\ A & \longmapsto & (\, w \perp_A c \;,\; W(A) \;,\; C(B)\,) \end{array}$$

defined by:

$* \perp_{at} *$	$:=$	at	unit	unit
$\langle x, u\rangle \perp_{A\wedge B} \langle y, v\rangle$	$:=$	$(x \perp_A y) \wedge (u \perp_B v)$	$W(A) \times W(B)$	$C(A) \times C(B)$
$\langle b^{\mathtt{nat}}, x, u\rangle \perp_{A\vee B} \langle y, v\rangle$	$:=$	$(x \perp_A y) \vee_{b^{\mathtt{nat}}} (u \perp_B v)$	$\mathbb{N} \times W(A) \times W(B)$	$C(A) \times C(B)$
$\langle f, F\rangle \perp_{A\to B} \langle x, v\rangle$	$:=$	$x \perp_A Fxv \to fx \perp_B v$	$W(A) \to W(B)$ $\times$ $W(A) \to C(B) \to C(A)$	$W(A) \times C(B)$
$f \perp_{\forall X\, x.A} \langle z, y\rangle$	$:=$	$fz \perp_{A\{x:=z\}} y$	$X \to W(A)$	$X \times C(A)$
$\langle z, u\rangle \perp_{\exists X\, x.A} y$	$:=$	$u \perp_{A\{x:=z\}} y$	$X \times W(A)$	$C(A)$

Example

$$\left.\begin{array}{l} \langle f, F\rangle \Vdash A \to B \\ w \Vdash A \end{array}\right\} \Longrightarrow \; fw \Vdash B$$

Remember that in Curry-Howard we only have $\mathrm{nf}(fw) \Vdash_{CH} B$.

Example

$$\left.\begin{array}{l} \langle f, F\rangle \Vdash A \to B \\ w \Vdash A \end{array}\right\} \Longrightarrow fw \Vdash B$$

Remember that in Curry-Howard we only have $\mathrm{nf}(fw) \Vdash_{CH} B$.

We have

$$\vdash \forall x^{W(A)}, v^{C(B)}.\ x \bot_A Fxv \to fx \bot_B v \qquad \text{and} \qquad \vdash \forall y^{C(A)}.\ w \bot_A y$$

We have to prove that

$$\vdash \forall c^{C(B)}.\ fw \bot_B c$$

Example

$$\left.\begin{array}{l}\langle f, F\rangle \Vdash A \to B \\ w \Vdash A\end{array}\right\} \implies fw \Vdash B$$

Remember that in Curry-Howard we only have $\mathrm{nf}(fw) \Vdash_{CH} B$.

We have

$$\vdash \forall x^{W(A)}, v^{C(B)}.\ x \bot_A Fxv \to fx \bot_B v \qquad \text{and} \qquad \vdash \forall y^{C(A)}.\ w \bot_A y$$

We have to prove that

$$\vdash \forall c^{C(B)}.\ fw \bot_B c$$

Intro c.
Apply the left one with $x := w$, and $v := c$.
Apply the right one with $y := Fxv$.
Apply the left to the right, and we are done.

Example

Let's compute the Dialectica interpretation of $\forall^X x \exists^Y y.\theta(x, y)$.

Example

Let's compute the Dialectica interpretation of $\forall^X x \exists^Y y.\theta(x,y)$.

$$W(\forall^X x \exists^Y y.\theta(x,y)) = X \to Y \times \text{unit}$$

$$C(\forall^X x \exists^Y y.\theta(x,y)) = X \times \text{unit}$$

$$\begin{aligned} f \perp_{\forall x \exists y.\theta(x,y)} z, * &= fz \perp_{\exists y.\theta(z,y)} * = (fz)_1, * \perp_{\exists y.\theta(z,y)} * \\ &= * \perp_{\theta(z,(fz)_1)} * \\ &= \theta(z,(fz)_1) \end{aligned}$$

So the Dialectica interpr. of $\forall x \exists y.\theta(x,y)$ is $\sim \forall x.\ \theta(x, fx)$, for fresh f.

Example

Let's compute the Dialectica interpretation of $\forall^X x \exists^Y y.\theta(x,y)$.

$$W(\forall^X x \exists^Y y.\theta(x,y)) = X \to Y \times \mathrm{unit}$$

$$C(\forall^X x \exists^Y y.\theta(x,y)) = X \times \mathrm{unit}$$

$$\begin{aligned} f \perp_{\forall x \exists y.\theta(x,y)} z, * &= fz \perp_{\exists y.\theta(z,y)} * = (fz)_1, * \perp_{\exists y.\theta(z,y)} * \\ &= * \perp_{\theta(z,(fz)_1)} * \\ &= \theta(z,(fz)_1) \end{aligned}$$

So the Dialectica interpr. of $\forall x \exists y.\theta(x,y)$ is $\sim \forall x.\ \theta(x, fx)$, for fresh f.

Corollary (of Adequacy)

*Every provably total function θ in HA is implementable in **T**:*

$\vdash_{HA} \forall^X x \exists^Y y.\ \theta(x,y) \implies$ *there is* $f \in \boldsymbol{T}$ *such that* $\vdash_{WE\text{-}HA^\omega} \forall^X x.\ \theta(x, fx)$.

Example

Let's compute the Dialectica interpretation of Markov's principle:

$$MP := \neg\forall^X x.\neg\theta(x) \to \exists^X x.\theta(x)$$

Example

Let's compute the Dialectica interpretation of Markov's principle:

$$MP := \neg\forall^X x.\neg\theta(x) \to \exists^X x.\theta(x)$$

$$W(MP) = \begin{array}{c}(\widetilde{X} \to X \times \mathrm{unit}) \\ \times \\ (\widetilde{X} \to \mathrm{unit} \to (X \to U) \times \mathrm{unit})\end{array} \simeq X \to X$$

$$C(MP) = \widetilde{X} \times \mathrm{unit} \simeq X$$

with $U := (\mathrm{unit} \to \mathrm{unit}) \times (\mathrm{unit} \to \mathrm{unit} \to \mathrm{unit}) \simeq \mathrm{unit}$ and
$\widetilde{X} := W(\neg\forall^X x.\neg\theta(x)) = ((X \to U) \to \mathrm{unit}) \times ((X \to U) \to \mathrm{unit} \to \mathrm{X} \times \mathrm{unit} \times \mathrm{unit}) \simeq X.$

Example

Let's compute the Dialectica interpretation of Markov's principle:

$$MP := \neg\forall^X x.\neg\theta(x) \to \exists^X x.\theta(x)$$

$$W(MP) = \begin{array}{c}(\widetilde{X} \to X \times \text{unit}) \\ \times \\ (\widetilde{X} \to \text{unit} \to (X \to U) \times \text{unit})\end{array} \simeq X \to X$$

$$C(MP) = \widetilde{X} \times \text{unit} \simeq X$$

with $U := (\text{unit} \to \text{unit}) \times (\text{unit} \to \text{unit} \to \text{unit}) \simeq \text{unit}$ and
$\widetilde{X} := W(\neg\forall^X x.\neg\theta(x)) = ((X \to U) \to \text{unit}) \times ((X \to U) \to \text{unit} \to \text{X} \times \text{unit} \times \text{unit}) \simeq X$.

$$\begin{array}{rcl}\langle f, *' \rangle \perp_{MP} \langle *'', \chi, * \rangle & = & \neg\neg(\ \xi_2 \perp_{\theta(\xi_1)} (\star_1 \xi_1)_2 \xi_2 \xi_3\ ) \to * \perp_{\theta((f\langle *, \chi\rangle)_1)} * \\ & = & \neg\neg(\ * \perp_{\theta(\xi_1)} *\ ) \to * \perp_{\theta((f\langle *, \chi\rangle)_1)} * \\ & = & \neg\neg\theta((\chi \star_1 \star_2)_1) \to \theta((f\langle *, \chi\rangle)_1)\end{array}$$

with $\star = *'\langle *'', \chi\rangle * : (X \to U) \times \text{unit}$ and $\xi := \chi \star_1 \star_2 : X \times (\text{unit} \times \text{unit})$.

A Few Examples

We can prove (constructively!) $\forall *'', \chi, * .\ \langle f, *' \rangle \perp_{MP} \langle *'', \chi, * \rangle$ choosing

$$f := \lambda x^{\widetilde{X}}.\ \langle (x_2(*_{X \to U})*)_1\ ,\ * \rangle \qquad \text{and} \qquad *' := \lambda\_,\_.\ \langle *_{X\to U}\ ,\ * \rangle$$

Theorem

Dialectica realises Markov's Principle: $\Vdash \neg\forall^X x.\neg\theta(x) \to \exists^X x.\theta(x)$.
Remember that Markov's Principle is not provable constructively!

$$W(MP) \quad = \quad \begin{array}{c}(\widetilde{X} \to X \times \text{unit}) \\ \times \\ (\widetilde{X} \to \text{unit} \to (X \to U) \times \text{unit})\end{array} \quad \simeq \quad X \to X$$

$$C(MP) \quad = \quad \widetilde{X} \times \text{unit} \quad \simeq \quad X$$

with $U := (\text{unit} \to \text{unit}) \times (\text{unit} \to \text{unit} \to \text{unit}) \simeq \text{unit}$ and
$\widetilde{X} := W(\neg\forall^X x.\neg\theta(x)) = ((X \to U) \to \text{unit}) \times ((X \to U) \to \text{unit} \to X \times \text{unit} \times \text{unit}) \simeq X$.

$$\begin{aligned}\langle f, *' \rangle \perp_{MP} \langle *'', \chi, * \rangle \quad &= \quad \neg\neg(\ \xi_2 \perp_{\theta(\xi_1)} (\star_1 \xi_1)_2 \xi_2 \xi_3\ ) \to * \perp_{\theta((f\langle *, \chi \rangle)_1)} * \\ &= \quad \neg\neg(\ * \perp_{\theta(\xi_1)} *\ ) \to * \perp_{\theta((f\langle *, \chi \rangle)_1)} * \\ &= \quad \neg\neg\theta((\chi \star_1 \star_2)_1) \to \theta((f\langle *, \chi \rangle)_1)\end{aligned}$$

with $\star = *'\langle *'', \chi \rangle * : (X \to U) \times \text{unit}$ and $\xi := \chi \star_1 \star_2 : X \times (\text{unit} \times \text{unit})$.

Hoare Triple: $A\langle f\rangle B$

Intuition: f : state $\to$ state and it "realises" $A \to B$

Hoare Triple: $A\langle f\rangle B$

Intuition: $f : \text{state} \to \text{state}$ and it "realises" $A \to B$ in the sense of the formula

$$\forall^{\text{state}} s.(\, A\{x := s\} \to B\{x := f s\}\,)$$

Hoare Triple: $A\langle f\rangle B$

Intuition: f : state $\rightarrow$ state and it "realises" $A \rightarrow B$ in the sense of the formula

$$\forall^{\text{state}} s. (\, A\{x := s\} \rightarrow B\{x := fs\}\,)$$

Theorem (Hoare Logic Soundness)

If the judgment $A\langle f\rangle B$ is derivable, then the formula above is provable (in some ambient theory, say WE-HA$^{\omega}$).

Hoare Triple: $A\langle f\rangle B$

Intuition: f : state $\rightarrow$ state and it "realises" $A \rightarrow B$ in the sense of the formula

$$\forall^{\text{state}} s.(\, A\{x := s\} \rightarrow B\{x := fs\}\,)$$

Theorem (Hoare Logic Soundness)

If the judgment $A\langle f\rangle B$ is derivable, then the formula above is provable (in some ambient theory, say WE-HA$^\omega$).

Say A and B are quantifier-free. Then the above formula is:

$$\forall^{\text{state}} s.(\, A\{x := s\} \rightarrow B\{x := fs\}\,)$$

Hoare Triple: $A\langle f\rangle B$

Intuition: f : state $\to$ state and it "realises" $A \to B$ in the sense of the formula

$$\forall^{\text{state}} s.(\, A\{x := s\} \to B\{x := fs\}\,)$$

Theorem (Hoare Logic Soundness)

If the judgment $A\langle f\rangle B$ is derivable, then the formula above is provable (in some ambient theory, say WE-HA$^\omega$).

Say A and B are quantifier-free. Then the above formula is:

$$\forall^{\text{state}} s.(\, * \perp_{A\{x:=s\}} * \to * \perp_{B\{x:=fs\}} * \,)$$

Hoare Triple: $A\langle f\rangle B$

Intuition: f : state $\to$ state and it "realises" $A \to B$ in the sense of the formula

$$\forall^{\text{state}} s.(\, A\{x := s\} \to B\{x := fs\}\,)$$

Theorem (Hoare Logic Soundness)

If the judgment $A\langle f\rangle B$ is derivable, then the formula above is provable (in some ambient theory, say WE-HA$^\omega$).

Say A and B are quantifier-free. Then the above formula is:

$$\forall^{\text{state}} s.(\ \langle s, *\rangle \perp_{(\exists^{\text{state}} x.A)} * \to \langle fs, *\rangle \perp_{(\exists^{\text{state}} x.B)} *\ )$$

Hoare Triple: $A\langle f\rangle B$

Intuition: $f : \text{state} \to \text{state}$ and it "realises" $A \to B$ in the sense of the formula

$$\forall^{\text{state}} s.(\, A\{x := s\} \to B\{x := fs\}\,)$$

Theorem (Hoare Logic Soundness)

If the judgment $A\langle f\rangle B$ is derivable, then the formula above is provable (in some ambient theory, say WE-HA$^\omega$).

Say A and B are quantifier-free. Then the above formula is:

$$\forall^{\text{state}} s.\ \langle f, *\rangle \perp_{(\exists^{\text{state}} x.A \to \exists^{\text{state}} x.B)} \langle s, *, *\rangle$$

Hoare Triple: $A\langle f\rangle B$

Intuition: f : state $\to$ state and it "realises" $A \to B$ in the sense of the formula

$$\forall^{\text{state}} s.(\, A\{x := s\} \to B\{x := fs\}\,)$$

Theorem (Hoare Logic Soundness)

If the judgment $A\langle f\rangle B$ is derivable, then the formula above is provable (in some ambient theory, say WE-HA$^\omega$).

Say A and B are quantifier-free. Then the above formula is:

$$\forall^{\text{state}} s.\ \langle f, *\rangle \perp_{(\exists^{\text{state}} x.A \to \exists^{\text{state}} x.B)} \langle s, *, *\rangle$$

Let's take this seriously in all its generality:

$$A\,\langle \mathtt{f} \,|\, \mathtt{F}\rangle\, B := \forall s, v.\ \langle \mathtt{f}, \mathtt{F}\rangle \perp_{A \to B} \langle s, v\rangle$$

for A, B any formulas. Intuition: $\langle \mathtt{f} \,|\, \mathtt{F}\rangle$ "realises" $A \to B$.

"Dialectica Hoare Logic"

Rules for deriving judgments $A \langle \mathtt{f} \mid \mathtt{F} \rangle B$, with $A, B \in$ WE-HA$^\omega$ and $\mathtt{f}, \mathtt{F} \in \mathbf{T}$

"Dialectica Hoare Logic"

Rules for deriving judgments $A \langle \mathtt{f} \mid \mathtt{F} \rangle B$, with $A, B \in \text{WE-HA}^\omega$ and $\mathtt{f}, \mathtt{F} \in \mathbf{T}$, such that

Theorem (Dialectica Hoare Logic Soundness)

If the judgment

$$A \langle \mathtt{f} \mid \mathtt{F} \rangle B$$

is derivable in DHL, then $\mathtt{f}, \mathtt{F}$ *realise* $A \to B$*, i.e.*

$$\vdash_{\mathit{WE\text{-}HA}^\omega} \quad \forall s\, v.\ \ s \perp_A \mathtt{F} s v \to \mathtt{f} s \perp_B v.$$

"Dialectica Hoare Logic"

Rules for deriving judgments $A \langle \mathtt{f} \,|\, \mathtt{F} \rangle B$, with $A, B \in \text{WE-HA}^\omega$ and $\mathtt{f}, \mathtt{F} \in \mathbf{T}$, such that

Theorem (Dialectica Hoare Logic Soundness)

If the judgment

$$A \langle \mathtt{f} \,|\, \mathtt{F} \rangle B$$

is derivable in DHL, then $\mathtt{f}, \mathtt{F}$ *realise* $A \to B$*, i.e.*

$$\vdash_{\mathit{WE\text{-}HA}^\omega} \quad \forall s\, v.\;\; s \perp_A \mathtt{F} s v \to \mathtt{f} s \perp_B v.$$

Usual Soundness Theorem by Gödel. But with the focus on programs $\mathtt{f}, \mathtt{F}$ and DHL as a specification system for them, instead of on formulas.

$$\bot\ \langle a \mid -\rangle\ P \qquad P\ \langle - \mid \alpha\rangle\ \top \qquad P\ \langle \mathtt{I} \mid \mathtt{proj}_2\rangle\ P \qquad \frac{P_\exists \to Q_\forall \in \mathrm{Ax}}{P_\exists\ \langle - \mid -\rangle\ Q_\forall} \qquad \frac{P_\exists\ \langle - \mid -\rangle\ Q_\forall}{P'_\exists\ \langle - \mid -\rangle\ Q'_\forall}\ \text{for}\ \frac{P_\exists \to Q_\forall}{P'_\exists \to Q'_\forall} \in \mathrm{Rule}$$

$$\frac{P\ \langle a, b \mid \alpha\rangle\ Q \wedge R}{P\ \langle b, a \mid \tilde\alpha\rangle\ R \wedge Q}\, p\wedge_R \qquad \frac{P \wedge Q\ \langle a \mid \alpha, \beta\rangle\ R}{Q \wedge P\ \langle \tilde a \mid \tilde\beta, \tilde\alpha\rangle\ R}\, p\wedge_L \qquad \frac{P\ \langle a, b \mid \alpha\rangle\ Q \vee_c R}{P\ \langle b, a \mid \tilde\alpha\rangle\ R \vee_{\bar c} Q}\, p\vee_R \qquad \frac{P \vee_c Q\ \langle a \mid \alpha, \beta\rangle\ R}{Q \vee_{\bar c} P\ \langle \tilde a \mid \tilde\beta, \tilde\alpha\rangle\ R}\, p\vee_L$$

$$\frac{P\ \langle a \mid \alpha\rangle\ Q}{P\ \langle a, b \mid \alpha_\pi\rangle\ Q \vee_0 R}\, \vee_R \qquad \frac{P\ \langle a \mid \alpha\rangle\ Q}{P \wedge R\ \langle a_\pi \mid \alpha_\pi, \beta\rangle\ Q}\, \wedge_L \qquad \frac{P\ \langle a, b \mid \alpha\rangle\ Q \wedge R}{P\ \langle a \mid \alpha_p\rangle\ Q}\, \wedge_R \qquad \frac{P \vee_0 R\ \langle a \mid \alpha, \beta\rangle\ Q}{P\ \langle a_p \mid \alpha_p\rangle\ Q}\, \vee_L$$

$$\frac{P \wedge \phi\ \langle a \mid \alpha\rangle\ R \quad Q \wedge \neg\phi\ \langle b \mid \beta\rangle\ R \quad \phi\ qf}{P \vee Q\ \langle \lambda x, y.\,\mathrm{if}\ \phi\ \mathrm{then}\ ax\ \mathrm{else}\ by \mid \alpha_\pi, \beta_\pi\rangle\ R}\, cond_L \qquad \frac{P\ \langle a \mid \alpha\rangle\ Q \quad P\ \langle b \mid \beta\rangle\ R}{P\ \langle a, b \mid \lambda x, v, w.\,\mathrm{if}\ |P|^x_{\alpha x v}\ \mathrm{then}\ \beta x w\ \mathrm{else}\ \alpha x v\rangle\ Q \wedge R}\, cond_R$$

$$\frac{P\ \langle a, b \mid \alpha\rangle\ Q \to R}{P \wedge Q\ \langle a \mid \alpha, b\rangle\ R}\, uncurry \qquad \frac{P \wedge Q\ \langle a \mid \alpha, \beta\rangle\ R}{P\ \langle a, \beta \mid \alpha\rangle\ Q \to R}\, curry \qquad \frac{P\ \langle a \mid \alpha\rangle\ Q \quad Q\ \langle b \mid \beta\rangle\ R}{P\ \langle \lambda x.b(a(x)) \mid \lambda x, w.\ \alpha x(\beta(ax)w)\rangle\ R}\, comp$$

$$\frac{P\ \langle a \mid \alpha\rangle\ Q(t)}{P\ \langle \lambda\_.t, a \mid \alpha\rangle\ \exists x\, Q(x)}\, \exists_R \qquad \frac{P(t)\ \langle a \mid \alpha\rangle\ Q}{\forall x\, P(x)\ \langle \lambda f.a(ft) \mid \lambda\_.t, \lambda f.\alpha(ft)\rangle\ Q}\, \forall_L$$

$$\frac{P(x)\ \langle a \mid \alpha\rangle\ Q}{\exists x\, P(x)\ \langle \lambda x.a \mid \lambda x.\alpha\rangle\ Q}\, \exists_L (x \notin Q) \qquad \frac{P\ \langle a \mid \alpha\rangle\ Q(x)}{P\ \langle \lambda y, x.ay \mid \lambda y, x.\alpha y\rangle\ \forall x\, Q(x)}\, \forall_R (x \notin P)$$

$$\frac{\exists x\, P(x)\ \langle a \mid \alpha\rangle\ Q}{P(t)\ \langle at \mid \alpha t\rangle\ Q}\, s_L \qquad \frac{P\ \langle a \mid \alpha\rangle\ \forall x\, Q(x)}{P\ \langle \lambda y.ayt \mid \lambda y, v.\alpha y t v\rangle\ Q(t)}\, s_R \qquad \frac{P_\forall\ \langle a, b \mid \alpha\rangle\ \exists x\, Q(x)}{P_\forall\ \langle b \mid \alpha\rangle\ Q(a)}\, \epsilon_R \qquad \frac{\forall x\, P_\forall(x)\ \langle - \mid \alpha, \beta\rangle\ Q_{qf}}{P_\forall(\alpha)\ \langle - \mid \beta\rangle\ Q_{qf}}\, \epsilon_L$$

$$\frac{P'\ \langle \mathtt{I} \mid \mathtt{proj}_2\rangle\ P \quad P\ \langle a \mid \alpha\rangle\ Q \quad Q\ \langle \mathtt{I} \mid \mathtt{proj}_2\rangle\ Q'}{P'\ \langle a \mid \alpha\rangle\ Q'}\, cons \qquad \frac{P\ \langle a \mid \alpha\rangle\ Q \quad a, \alpha = b, \beta}{P\ \langle b \mid \beta\rangle\ Q}\, ext \qquad \frac{P(x)\ \langle a(x) \mid \alpha(x)\rangle\ P(x+1)}{P(0)\ \langle \mathrm{rec}\ a \mid \mathrm{rec}^*\ a\, \alpha\rangle\ \forall x.\ P(x)}\, ind$$

Paper and to go further

Paper and to go further

Paper The procedural view of Dialectica is compatible with the addition of a well-founded `while` in the extraction language.

Paper and to go further

Paper The procedural view of Dialectica is compatible with the addition of a well-founded `while` in the extraction language.

Q. Is Dialectica compatible with the addition of a **variable allocation instruction?** ⇒ Extract **imperative** programs!

Paper and to go further

Paper The procedural view of Dialectica is compatible with the addition of a well-founded `while` in the extraction language.

Q. Is Dialectica compatible with the addition of a **variable allocation instruction?** $\Rightarrow$ Extract **imperative** programs!

Paper We show that a fragment of the language admits a Hoare-Logic-like semantics, and a procedural big-step operational semantics.

Paper The procedural view of Dialectica is compatible with the addition of a well-founded `while` in the extraction language.

Q. Is Dialectica compatible with the addition of a **variable allocation instruction?** $\Rightarrow$ Extract **imperative** programs!

Paper We show that a fragment of the language admits a Hoare-Logic-like semantics, and a procedural big-step operational semantics.

Q. The following rule is admissible in DHL:

$$\frac{P_1\,\langle a\,|\,\alpha\rangle\,Q_1 \quad P_2\,\langle b\,|\,\beta\rangle\,Q_2}{P_1 \wedge P_2\ \ \langle a, b\,|\,\alpha, \beta\rangle\ \ Q_1 \wedge Q_2}\ \textit{Where } a, \alpha \textit{ and } b, \beta \textit{ operate on disjoint variables}$$

So, **frame rule?**
$$\frac{P_1\,\langle a\,|\,\alpha\rangle\,Q_1 \quad P_2\,\langle b\,|\,\beta\rangle\,Q_2}{P_1 * P_2\ \ \langle a, \alpha\rangle \,||\, \langle b, \beta\rangle\ \ Q_1 * Q_2}$$

Also: **Dialectica for Bunched/Separation Logic?**

Paper and to go further

Paper The procedural view of Dialectica is compatible with the addition of a well-founded `while` in the extraction language.

Q. Is Dialectica compatible with the addition of a **variable allocation instruction?** $\Rightarrow$ Extract **imperative** programs!

Paper We show that a fragment of the language admits a Hoare-Logic-like semantics, and a procedural big-step operational semantics.

Q. The following rule is admissible in DHL:

$$\frac{P_1\,\langle a\,|\,\alpha\rangle\,Q_1 \quad P_2\,\langle b\,|\,\beta\rangle\,Q_2}{P_1 \wedge P_2\ \ \langle a, b\,|\,\alpha, \beta\rangle\ \ Q_1 \wedge Q_2}\ \textit{Where } a, \alpha \textit{ and } b, \beta \textit{ operate on disjoint variables}$$

So, **frame rule?** $$\frac{P_1\,\langle a\,|\,\alpha\rangle\,Q_1 \quad P_2\,\langle b\,|\,\beta\rangle\,Q_2}{P_1 * P_2\ \ \langle a, \alpha\rangle \,||\, \langle b, \beta\rangle\ \ Q_1 * Q_2}$$

Also: **Dialectica for Bunched/Separation Logic?**

Paper We hint some rules combining Dialectica and double negation translation, for handling classical logic.

Paper The procedural view of Dialectica is compatible with the addition of a well-founded `while` in the extraction language.

Q. Is Dialectica compatible with the addition of a **variable allocation instruction?** $\Rightarrow$ Extract **imperative** programs!

Paper We show that a fragment of the language admits a Hoare-Logic-like semantics, and a procedural big-step operational semantics.

Q. The following rule is admissible in DHL:

$$\frac{P_1\,\langle a\,|\,\alpha\rangle\,Q_1 \qquad P_2\,\langle b\,|\,\beta\rangle\,Q_2}{P_1 \wedge P_2\ \ \langle a, b\,|\,\alpha, \beta\rangle\ \ Q_1 \wedge Q_2}\ \textit{Where } a, \alpha \textit{ and } b, \beta \textit{ operate on disjoint variables}$$

So, **frame rule?**
$$\frac{P_1\,\langle a\,|\,\alpha\rangle\,Q_1 \qquad P_2\,\langle b\,|\,\beta\rangle\,Q_2}{P_1 * P_2\ \ \langle a, \alpha\rangle \,||\, \langle b, \beta\rangle\ \ Q_1 * Q_2}$$

Also: **Dialectica for Bunched/Separation Logic?**

Paper We hint some rules combining Dialectica and double negation translation, for handling classical logic.

Q. Does this give a realisability stronger then Krivine one?
Also, Libraries for **Computer-Assisted** Dialectica realisers **extraction**?

Paper The procedural view of Dialectica is compatible with the addition of a well-founded `while` in the extraction language.

Q. Is Dialectica compatible with the addition of a **variable allocation instruction?** $\Rightarrow$ Extract **imperative** programs!

Paper We show that a fragment of the language admits a Hoare-Logic-like semantics, and a procedural big-step operational semantics.

Q. The following rule is admissible in DHL:

$$\frac{P_1 \langle a \,|\, \alpha \rangle Q_1 \quad P_2 \langle b \,|\, \beta \rangle Q_2}{P_1 \wedge P_2 \;\; \langle a, b \,|\, \alpha, \beta \rangle \;\; Q_1 \wedge Q_2} \; \textit{Where } a, \alpha \textit{ and } b, \beta \textit{ operate on disjoint variables}$$

So, **frame rule?**
$$\frac{P_1 \langle a \,|\, \alpha \rangle Q_1 \quad P_2 \langle b \,|\, \beta \rangle Q_2}{P_1 * P_2 \;\; \langle a, \alpha \rangle \,||\, \langle b, \beta \rangle \;\; Q_1 * Q_2}$$

Also: **Dialectica for Bunched/Separation Logic?**

Paper We hint some rules combining Dialectica and double negation translation, for handling classical logic.

Q. Does this give a realisability stronger then Krivine one?
Also, Libraries for **Computer-Assisted** Dialectica realisers **extraction**?

Thank you!

Update WE-HA$^\omega$

Update WE-HA$^\omega$

- Term PL: $\cdots \mid \prec : X \to X \to \mathtt{nat}$
  $\mid \mathtt{whilerec}_{\phi,a} : (X \to U) \to (X \to U \to U) \to X \to U$
- Formulas: same as before
- Axioms: same as before + the following for $\phi\{x\}$ q.f.:

$$(\phi\{x := y\} \to ay \prec y) \to$$
$$\mathtt{whilerec}_{\phi,a}\, u\, F\, y =_U \mathtt{if}\ \phi\{x := y\}\ \mathtt{then}\ F\, y\, (\mathtt{whilerec}_{\phi,a}\, u\, F\, (ay))\ \mathtt{else}\ (uy)$$

- Rules: same as before + $\dfrac{\forall x. (\, (\forall y \prec x. A\{x := y\}) \to A\,)}{\forall x. A}$ *(w.-f. induction)*

Update WE-HA$^\omega$

- Term PL: $\cdots \mid \prec : X \to X \to \mathtt{nat}$
  $\mid \mathtt{whilerec}_{\phi,a} : (X \to U) \to (X \to U \to U) \to X \to U$
- Formulas: same as before
- Axioms: same as before + the following for $\phi\{x\}$ q.f.:

$$(\phi\{x := y\} \to ay \prec y) \to$$
$$\mathtt{whilerec}_{\phi,a}\, u\, F\, y =_U \mathtt{if}\ \phi\{x := y\}\ \mathtt{then}\ F\, y\, (\mathtt{whilerec}_{\phi,a}\, u\, F\, (ay))\ \mathtt{else}\ (uy)$$

- Rules: same as before + $\dfrac{\forall x. \left((\forall y \prec x. A\{x := y\}) \to A \right)}{\forall x. A}$ *(w.-f. induction)*

Remark

The sugars

$$\begin{array}{rcllll} \mathtt{while}\, \phi\, \mathtt{do}\, a & := & \mathtt{whilerec}_{\phi,a} & \mathtt{I} & \mathtt{proj}_2 & : X \to X \\ \mathtt{while}^*\, \phi\, \mathtt{do}\, (a, \alpha) & := & \mathtt{whilerec}_{\phi,a} & \mathtt{proj}_2 & (\lambda x, f, v.\, \alpha x(fv)) & : X \to V \to V \end{array}$$

behave in WE-HA$^\omega$ like a usual *well-founded* while and a backward while, resp.

Dialectica with While

Add to DHL the rule:

$$\frac{\exists x.\,(P_\forall(x) \wedge \phi(x))\ \langle a \,|\, \alpha \rangle\ \exists x.P_\forall(x) \qquad \forall x.\,(\phi(x) \to ax \prec x)}{\exists x.P_\forall(x)\,\langle\ \texttt{while}\,\phi\ \texttt{do}\ a \,|\, \texttt{while}^*\,\phi\,\texttt{do}\,(a,\alpha)\rangle\ \exists x.\,(P_\forall(x) \wedge \neg\phi(x))}$$

Theorem

Dialectica Hoare Logic Soundness keeps holding, i.e. the $\texttt{f}, \texttt{F}$ *(now with possible loops) in* $A\,\langle \texttt{f} \,|\, \texttt{F}\rangle\, B$ *still realise* $A \to B$ *(now with well-founded induction).*

A minimum principle: $\exists x.\,\theta \vdash \exists x.\,(\theta \wedge \forall y \prec x.\neg\,\theta(y))$

with $\prec$ well-founded and $\theta\{x^X\}$ quantifier-free.

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{}{\theta \wedge \phi_g \,\langle - \,|\, - \rangle\, \theta(gx)}
      }{\theta \wedge \phi_g \,\langle gx \,|\, - \rangle\, \exists x.\,\theta}\;\exists_R
    }{\exists x.\,(\theta \wedge \phi_g)\,\langle g \,|\, - \rangle\, \exists x.\,\theta}\;\exists_L
    \qquad \forall x.\,(\phi_g \to gx \prec x)
  }{\exists x.\,\theta\,\langle \texttt{while}\;\phi_g\;\texttt{do}\;g \,|\, - \rangle\, \exists x.\,(\theta \wedge \neg\phi_g)}\;\mathit{while}
}{
  \dfrac{\exists x.\,\theta\,\langle \lambda x, g.(\texttt{while}\;\phi_g\;\texttt{do}\;g)x \,|\, - \rangle\, \forall g \exists x.\,(\theta \wedge \neg\phi_g)}{\exists x.\,\theta\,\langle \lambda x, g.(\texttt{while}\;\phi_g\;\texttt{do}\;g)x \,|\, - \rangle\, \neg\neg\exists x.\,(\theta \wedge \forall y \prec x.\,\neg\theta(y))}\;N
}\;\forall_R
$$

with $\phi_g := gx \prec x \wedge \theta(gx)$.

Idea: trial-and-error. (Appears very often in proof mining).

Fix fresh sets of commands $\overrightarrow{Comm}$, $\overleftarrow{Comm}$ of new types $S \to S$ and $S \to T \to T$.
$\mathrm{LOOP}_D :=$ IMP with commands from above and *without* variable allocation:

$$C ::= \texttt{skip} \mid \langle c \mid \gamma \rangle \mid C; C \mid \texttt{if}\ \phi\ \texttt{then}\ C\ \texttt{else}\ C \mid \texttt{while}\ \phi\ \texttt{do}\ C$$

Fix fresh sets of commands $\overrightarrow{Comm}, \overleftarrow{Comm}$ of new types $S \to S$ and $S \to T \to T$. $\mathrm{LOOP}_D :=$ IMP with commands from above and *without* variable allocation:

$$C ::= \texttt{skip} \mid \langle c \,|\, \gamma \rangle \mid C; C \mid \texttt{if}\ \phi\ \texttt{then}\ C\ \texttt{else}\ C \mid \texttt{while}\ \phi\ \texttt{do}\ C$$

Define a translation $\mathrm{LOOP}_D \to \mathbf{T}^{S \to S} \times \mathbf{T}^{S \to T \to T}$ as:

LOOP_D	$(\_)^+$	$(\_)^-$
$\langle c \,\|\, \gamma \rangle$	c	γ
$\texttt{skip}$	$\texttt{I}$	$\texttt{proj}_2$
$C_1; C_2$	$\lambda x.\, C_2^+(C_1^+ x)$	$\lambda x, w.\ C_1^- x (C_2^- (C_1^+ x) w)$
$\texttt{if}\ \phi\ \texttt{then}\ C_1\ \texttt{else}\ C_2$	$\lambda s.\texttt{if}\ \phi(s)\ \texttt{then}\ C_1^+ s\ \texttt{else}\ C_2^+ s$	$\lambda s, t.\texttt{if}\ \phi(s)\ \texttt{then}\ C_1^- st\ \texttt{else}\ C_2^- st$
$\texttt{while}\ \phi\ \texttt{do}\ C$	$\texttt{while}\ \phi\ \texttt{do}\ C^+$	$(\texttt{while}^*\ \phi\, \texttt{do}\, C^+), C^-$

“Hoare Logic” for LOOP$_D$

$$\frac{}{[P]\,\texttt{skip}\,[P]} \qquad \frac{P(s,\gamma st) \to Q(cs,t) \in \mathrm{Ax}}{[P]\,\langle c \,|\, \gamma\rangle\,[Q]} \qquad \frac{[P]\,C_1\,[Q] \quad [Q]\,C_2\,[R]}{[P]\,C_1; C_2\,[R]}$$

$$\frac{[P \wedge \phi(s)]\,C_1\,[R] \quad [Q \wedge \neg\phi(s)]\,C_2\,[R]}{[P \vee_{\phi(s)} Q]\,\texttt{if}\;\phi(s)\;\texttt{then}\;C_1\;\texttt{else}\;C_2\,[R]} \qquad \frac{[P \wedge \phi(s)]\,C\,[P] \quad \phi(s)(s) \to C^+s \prec s}{[P]\,\texttt{while}\;\phi(s)\;\texttt{do}\;C\,[P \wedge \neg\phi(s)]}$$

$$\frac{P' \to P \quad [P]\,C\,[Q] \quad Q \to Q'}{[P']\,C\,[Q']}$$

“Hoare Logic” for LOOP_D

$$\frac{}{[P]\,\texttt{skip}\,[P]} \qquad \frac{P(s,\gamma st) \to Q(cs,t) \in \mathrm{Ax}}{[P]\,\langle c \,|\, \gamma\rangle\,[Q]} \qquad \frac{[P]\,C_1\,[Q] \quad [Q]\,C_2\,[R]}{[P]\,C_1; C_2\,[R]}$$

$$\frac{[P \wedge \phi(s)]\,C_1\,[R] \quad [Q \wedge \neg\phi(s)]\,C_2\,[R]}{[P \vee_{\phi(s)} Q]\,\texttt{if}\ \phi(s)\ \texttt{then}\ C_1\ \texttt{else}\ C_2\,[R]} \qquad \frac{[P \wedge \phi(s)]\,C\,[P] \quad \phi(s)(s) \to C^+ s \prec s}{[P]\,\texttt{while}\ \phi(s)\ \texttt{do}\ C\,[P \wedge \neg\phi(s)]}$$

$$\frac{P' \to P \quad [P]\,C\,[Q] \quad Q \to Q'}{[P']\,C\,[Q']}$$

Theorem (Soundness wrt Dialectica)

Restrict the above rules to quantifier-free formulas with only one variable s^S and one t^T.
If $[P]\,C\,[Q]$ is derivable then DHL derives $\exists^S s \forall^T t.P\ \langle C^+ \,|\, C^-\rangle\ \exists^S s \forall^T t.Q$ and so

$$\vdash_{\mathit{WE\text{-}HA}^\omega} \forall s,t.\,\big(P\{t := C^- st\} \to Q\{s := C^+ s\}\big)$$

Big-step Operational semantics of LOOP_D

Big-step Operational semantics of LOOP$_D$

Forward OS: $\mathbf{T}^S \times \mathbf{LOOP}_D \vec{\Downarrow} \ \mathbf{T}^S \times (\mathbf{T}^S)^* \times (\mathbf{T}^{S \to T \to T})^*$

$$\frac{}{s, \texttt{skip} \vec{\Downarrow} s, [\,], [\,]} \qquad \frac{}{s, \langle c \mid \gamma \rangle \vec{\Downarrow} cs, s :: [\,], \gamma :: [\,]} \qquad \frac{s, C_1 \vec{\Downarrow} s', \sigma, \Gamma \quad s', C_2 \vec{\Downarrow} s'', \sigma', \Gamma'}{s, C_1; C_2 \vec{\Downarrow} s'', \sigma' :: \sigma, \Gamma' :: \Gamma}$$

$$\frac{\phi(s) \quad s, C_1 \vec{\Downarrow} s', \sigma, \Gamma}{s, \text{if } \phi \text{ then } C_1 \text{ else } C_2 \vec{\Downarrow} s', \sigma, \Gamma} \qquad \frac{\neg\phi(s) \quad s, C_2 \vec{\Downarrow} s', \sigma, \Gamma}{s, \text{if } \phi \text{ then } C_1 \text{ else } C_2 \vec{\Downarrow} s', \sigma, \Gamma}$$

$$\frac{\neg\phi(s)}{s, \texttt{while}\ \phi\ \texttt{do}\ C \vec{\Downarrow} s, [\,], [\,]} \qquad \frac{\phi(s) \quad s, C \vec{\Downarrow} s', \sigma, \Gamma \quad s' \prec s \quad s', \texttt{while}\ \phi\ \texttt{do}\ C \vec{\Downarrow} s'', \sigma', \Gamma'}{s, \texttt{while}\ \phi\ \texttt{do}\ C \vec{\Downarrow} s'', \sigma' :: \sigma, \Gamma' :: \Gamma}$$

Big-step Operational semantics of LOOP_D

Forward OS: $\mathbf{T}^S \times \mathbf{LOOP}_D \;\vec{\Downarrow}\; \mathbf{T}^S \times (\mathbf{T}^S)^* \times (\mathbf{T}^{S\to T\to T})^*$

$$\frac{}{s, \texttt{skip} \vec{\Downarrow} s, [\,], [\,]} \qquad \frac{}{s, \langle c \,|\, \gamma\rangle \vec{\Downarrow} cs, s :: [\,], \gamma :: [\,]} \qquad \frac{s, C_1 \vec{\Downarrow} s', \sigma, \Gamma \quad s', C_2 \vec{\Downarrow} s'', \sigma', \Gamma'}{s, C_1; C_2 \vec{\Downarrow} s'', \sigma' :: \sigma, \Gamma' :: \Gamma}$$

$$\frac{\phi(s) \quad s, C_1 \vec{\Downarrow} s', \sigma, \Gamma}{s, \texttt{if } \phi \texttt{ then } C_1 \texttt{ else } C_2 \vec{\Downarrow} s', \sigma, \Gamma} \qquad \frac{\neg\phi(s) \quad s, C_2 \vec{\Downarrow} s', \sigma, \Gamma}{s, \texttt{if } \phi \texttt{ then } C_1 \texttt{ else } C_2 \vec{\Downarrow} s', \sigma, \Gamma}$$

$$\frac{\neg\phi(s)}{s, \texttt{while } \phi \texttt{ do } C \vec{\Downarrow} s, [\,], [\,]} \qquad \frac{\phi(s) \quad s, C \vec{\Downarrow} s', \sigma, \Gamma \quad s' \prec s \quad s', \texttt{while } \phi \texttt{ do } C \vec{\Downarrow} s'', \sigma', \Gamma'}{s, \texttt{while } \phi \texttt{ do } C \vec{\Downarrow} s'', \sigma' :: \sigma, \Gamma' :: \Gamma}$$

Backward OS: $(\mathbf{T}^S)^* \times (\mathbf{T}^{S\to T\to T})^* \times \mathbf{T}^T \;\overleftarrow{\Downarrow}\; (\mathbf{T}^S)^* \times (\mathbf{T}^{S\to T\to T})^* \times \mathbf{T}^T$

$$\frac{}{\sigma, \Gamma, t \overleftarrow{\Downarrow} \sigma, \Gamma, t} \qquad \frac{}{s :: \sigma, \gamma :: \Gamma, t \overleftarrow{\Downarrow} \sigma, \Gamma, \gamma st} \qquad \frac{\sigma, \Gamma, t \overleftarrow{\Downarrow} \sigma', \Gamma', t' \quad \sigma', \Gamma', t' \overleftarrow{\Downarrow} \sigma'', \Gamma'', t''}{\sigma, \Gamma, t \overleftarrow{\Downarrow} \sigma'', \Gamma'', t''}$$

Big-step Operational semantics of LOOP_D

Theorem (Forward+Backward OS = Backpropagation in LOOP_D)

Restrict $LOOP_D$ to only terminating (in WE-HA$^\omega$) while loops.
Let C be a command. Then:
For all $s : S$ there exist $\sigma : S^*$ and $\Gamma : (S \to T \to T)^*$ such that

1. $$s, C \overrightarrow{\Downarrow} (C^+ s), \sigma, \Gamma$$
2. *for all $t : T$,*
$$\sigma, \Gamma, t \overleftarrow{\Downarrow} [\,], [\,], (C^- s t).$$

Dialectica can be used to implement high-order backpropagation ! (already hinted by Kerjean, Pédrot)