An overview about Dialectica as Differentiation

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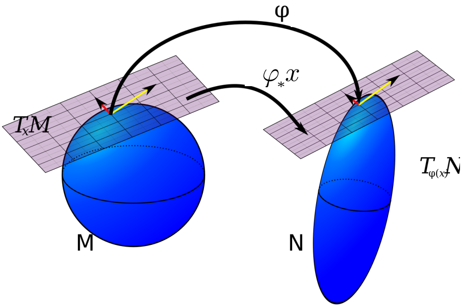
Differentiation



Differentiation

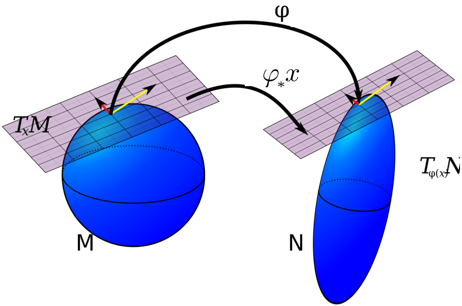
Cartesian differential categories (∼'09)

$$\frac{f : A \to B}{Df : A \times A \to B}$$

Cartesian tangent categories ('14)

$$\frac{f : A \to B}{Tf : TA \to TB}$$

Forward



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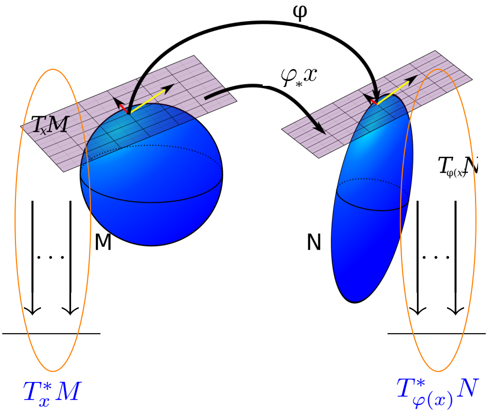

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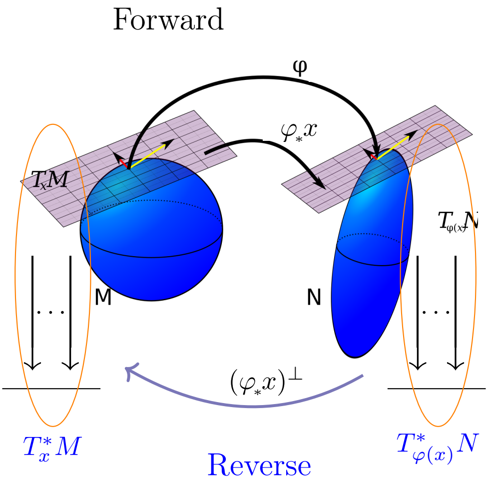

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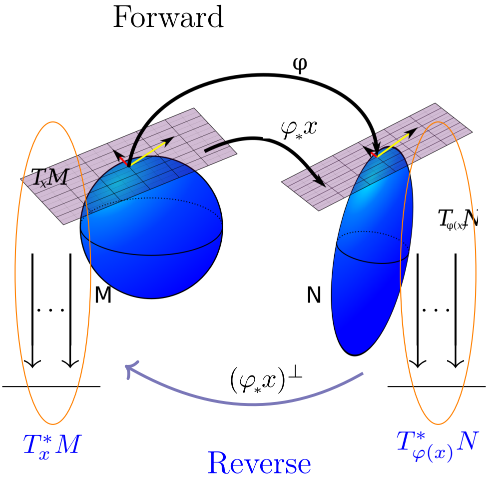
Cartesian reverse diff. categories ('20)

$$\frac{f : A \to B}{Rf : A \times B \to A}$$

Cart. reverse tangent categories ('24)

$$\frac{f : A \to B}{T^*f : f^*T^*B \to T^*A}$$



Dialectica (overview)

	Source $\to$ Target
Gödel ('58)	$A \in \mathrm{HA} \quad \longmapsto \quad A_D\{w, c\} \in \mathrm{T}$ *such that* $\vdash_{\mathrm{HA}} A \quad \Longrightarrow \quad \vdash_{\mathrm{T}} A_D\{\mathtt{M}, c\}$ *for some* $\mathtt{M} \in \mathrm{T}$

Dialectica (overview)

Definition 1 (Dialectica interpretation). *For each formula A of intuitionistic logic we associate a new quantifier-free formula $A_D(\boldsymbol{x};\boldsymbol{y})$ inductively as follows:*

$$(A_{\mathsf{at}})^D :\equiv A_{\mathsf{at}}, \quad \textit{when } A_{\mathsf{at}} \textit{ is an atomic formula.}$$

Assume we have already defined $A_D(\boldsymbol{x};\boldsymbol{y})$ and $B_D(\boldsymbol{v};\boldsymbol{w})$. We then define

$$\begin{aligned}
(A \wedge B)_D(\boldsymbol{x},\boldsymbol{v};\boldsymbol{y},\boldsymbol{w}) &:\equiv A_D(\boldsymbol{x};\boldsymbol{y}) \wedge B_D(\boldsymbol{v};\boldsymbol{w}) \\
(A \vee B)_D(\boldsymbol{x},\boldsymbol{v},z;\boldsymbol{y},\boldsymbol{w}) &:\equiv A_D(\boldsymbol{x};\boldsymbol{y}) \diamond_z B_D(\boldsymbol{v};\boldsymbol{w}) \\
(A \to B)_D(\boldsymbol{f},\boldsymbol{g};\boldsymbol{x},\boldsymbol{w}) &:\equiv A_D(\boldsymbol{x};\boldsymbol{f}\boldsymbol{w}\boldsymbol{x}) \to B_D(\boldsymbol{g}\boldsymbol{x};\boldsymbol{w}) \\
(\forall z A)_D(\boldsymbol{f};\boldsymbol{y},z) &:\equiv A_D(\boldsymbol{f}z;\boldsymbol{y}) \\
(\exists z A)_D(\boldsymbol{x},z;\boldsymbol{y}) &:\equiv A_D(\boldsymbol{x};\boldsymbol{y}).
\end{aligned}$$

Finally, we define $(A)^D :\equiv \exists\boldsymbol{x}\forall\boldsymbol{y}A_D(\boldsymbol{x};\boldsymbol{y})$.

Dialectica (overview)

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		such that	
	$\vdash_{\mathrm{HA}} A$	$\Longrightarrow$	$\vdash_{\mathrm{T}} A_D\{\mathtt{M}, c\}$ *for some* $\mathtt{M} \in \mathrm{T}$
Pédrot ('15)	$A \in \Lambda$	$\longmapsto$	$W(A), C(A) \in \mathbf{P}$
	$\mathtt{M} \in \Lambda$	$\longmapsto$	$\mathtt{M}^\bullet, \mathtt{M}_\mathtt{x} \in \mathbf{P}$ *(for* $\boldsymbol{x}$ *variable)*
		such that	
	$\mathtt{x} : A \vdash_\Lambda \mathtt{M} : B$	$\Longrightarrow$	$\begin{cases} \mathtt{x} : W(A) \vdash_{\mathbf{P}} \mathtt{M}^\bullet : W(B) \\ \mathtt{x} : W(A) \vdash_{\mathbf{P}} \mathtt{M}_\mathtt{x} : C(B) \to \mathcal{M}[C(A)] \end{cases}$

Dialectica (Transformation)

	α	$E \to F$
W	α_W	$W(E) \to W(F)$ $\times$ $W(E) \times C(F) \to \mathcal{M}[C(E)]$
C	α_C	$W(E) \times C(F)$

	$\mathtt{x}$	$\lambda\mathtt{x.M}$	$\mathtt{PQ}$
$(\_)^\bullet$	$\mathtt{x}$	$\left\langle \begin{array}{c} \lambda\mathtt{x.M}^\bullet \\ \lambda\pi.(\lambda\mathtt{x.M_x})\pi^1\pi^2 \end{array} \right\rangle$	$\mathtt{P}^{\bullet 1}\mathtt{Q}^\bullet$
$(\_)_\mathtt{y}$	$\begin{cases} \lambda\pi.[\pi], & \mathtt{x} = \mathtt{y} \\ \lambda\pi.0, & \mathtt{y} \neq \mathtt{y} \end{cases}$	$\lambda\pi.(\lambda\mathtt{x.M_y})\pi^1\pi^2$	$\lambda\pi. \left(\begin{array}{c} \mathtt{P_y}\langle \mathtt{Q}^\bullet, \pi\rangle \\ + \\ \mathtt{P}^{\bullet 2}\langle \mathtt{Q}^\bullet, \pi\rangle \gg\!= \mathtt{Q}_y \end{array} \right)$

A model $(\mathcal{C}, !)$ of Classical Differential Linear Logic

Arrows in $\mathcal{C}$*:* $A \overset{f}{\multimap} B$ *(linear)* *Arrows in* $\mathcal{C}_!$*:* $A \overset{f}{\to} B := !A \overset{f}{\multimap} B$ *(non-linear)*

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Cartesian
+SMC
+Seely

$$\frac{A \quad B}{A \& B} \qquad \frac{A \quad B}{A \otimes B} \qquad \frac{}{!\top}$$

$$\frac{}{\mathrm{ev}_{A,B} : [A \multimap B] \otimes A \multimap B}$$

$$\frac{f : A \multimap [E \multimap F]}{\lambda f : A \otimes E \multimap F}$$

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comm. monoids enriched

$$\frac{}{0_{A,B} : A \multimap B} \qquad \frac{f : A \multimap B \quad g : A \multimap B}{f + g : A \multimap B}$$

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DiLL magic

$$\frac{f : A \to B}{\partial f : A \otimes !A \multimap B} \qquad \frac{\dfrac{f : A \to B}{f_* : A \to [A \multimap B]} \quad a :!\top \multimap A}{f_* a : A \multimap B}$$

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Cartesian +SMC +Seely	$\dfrac{A \quad B}{A\&B}$ $\dfrac{A \quad B}{A \otimes B}$ $\dfrac{}{!\top}$	$\dfrac{}{\mathrm{ev}_{A,B} : [A \multimap B] \otimes A \multimap B}$ $\dfrac{f : A \multimap [E \multimap F]}{\lambda f : A \otimes E \multimap F}$
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$\mathcal{C}_!$ is a model of differential λ-calculus where we can transpose linear arrows:

$$\frac{f : A \to B}{Df : A \times A \to B} \quad \textit{(in } \mathcal{C}_!,\ E \times F := E\&F \textit{)}$$

Dialectica and (Categorical) Differentiation

$$\sim_B \quad \subseteq \quad \{\vdash_{\mathbf{P}} \mathtt{M} : W(B)\} \quad \times \quad \mathcal{C}_!(\top, B)$$

$$\bowtie_B^A \quad \subseteq \quad \{\vdash_{\mathbf{P}} \mathtt{M} : C(B) \to \mathcal{M}[C(A)]\} \quad \times \quad \mathcal{C}_!(A, B) \times \mathcal{C}(B^\perp, A^\perp)$$

$\mathtt{M} \sim_{E \to F} f$	*for all* $\mathtt{H} \sim_E e$, *we have* $\quad \mathtt{M}^1\mathtt{H} \sim_F f\big\vert_e : F$ *and* $\quad \lambda\pi.\mathtt{M}^2\langle\mathtt{H}, \pi\rangle \bowtie_F^E \begin{pmatrix} \lambda^{-1} f : E \to F \\ ((\lambda^{-1} f)_* e)^\perp : F^\perp \multimap E^\perp \end{pmatrix}$
$\mathtt{M} \bowtie_{E \to F}^A \binom{f}{g}$	*for all* $\mathtt{H} \sim_E e$, *we have* $\lambda\pi.\mathtt{M}\langle\mathtt{H}, \pi\rangle \bowtie_F^A \begin{pmatrix} f\big\vert_e : A \to F \\ g^\perp\big\vert_e^\perp : F^\perp \multimap A^\perp \end{pmatrix}$

The theorem

Let $\mathtt{x} : A \vdash_\Lambda \mathtt{M} : B$. Then:

$$1) \quad (\lambda \mathtt{x}.\mathtt{M})^\bullet \quad \sim_{A \to B} \quad [\![\lambda \mathtt{x}.\mathtt{M}]\!] : [A \to B]$$

$$2) \quad (\lambda \mathtt{x}.\mathtt{M}_\mathtt{x})\mathtt{N} \quad \bowtie_B^A \quad \begin{pmatrix} [\![\mathtt{M}]\!] & : & A \to B \\ ([\![\mathtt{M}]\!]_* a)^\perp & : & B^\perp \multimap A^\perp \end{pmatrix} \quad \textit{for all } \mathtt{N} \sim_A a.$$

Moral:

$$(\lambda \mathtt{x}.\mathtt{M}^\bullet \, , \, \lambda \mathtt{x}.\mathtt{M}_\mathtt{x})$$

"represents" the pair $([\![\mathtt{M}]\!] \, , \, R[\![\mathtt{M}]\!])$, where

$$R[\![\mathtt{M}]\!] : A \times B^\perp \to A^\perp$$

is the reverse differential of $[\![\mathtt{M}]\!]$.

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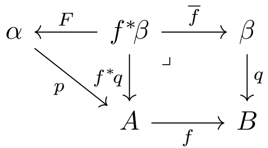
  Not really, we are just formulating it differently.
- Is this correspondence astonishing/magic? Can we find some "reason" clarifying it?
  Definitely yes at first sight... but then we can clearly understand its reason by looking at the categorical framework behind it.

Lens Categories

The category Lens($\mathcal{L}$) of lenses over $\mathcal{L}$ is defined as follows:

- objects: arrows in $\mathcal{L}$, which we think as fibre bundles and we write $p : \binom{\alpha}{A}$
- arrows from $p : \binom{\alpha}{A}$ to $q : \binom{\beta}{B}$ are the data of both a $f : A \to B$ in $\mathcal{L}$ and a span $\alpha \xleftarrow{F} f^*\beta \xrightarrow{\overline{f}} \beta$ in $\mathcal{L}$, taken from the following pullback diagram:



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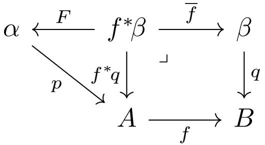

Let $\mathcal{E}\mathrm{Lens}(\mathcal{L})$ be the full subcategory of $\mathrm{Lens}(\mathcal{L})$ of trivial bundles, i.e. first projections. Concretely:

- Objects are first projections $\pi_1 : \binom{A \times X}{A}$
- An arrow from $\pi_1 : \binom{A \times X}{A}$ to $\pi_1 : \binom{B \times Y}{B}$ is given by an $f : A \to B$ and a span $A \times X \xleftarrow{F} A \times Y \xrightarrow{f \times 1} B \times Y$ such that $F ; \pi_1^{A,X} = \pi_1^{A,Y}$.

Differentiation through the lens of lenses

Let $\mathcal{L}$ be a Cartesian (closed, if you want λ-calculus) differential category where from the differential Df of a function f (a primitive data in $\mathcal{L}$) we can define the reverse differential Rf of f. (Think of $\mathcal{L} := \mathcal{C}_!$ of the first part).

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We have a functor $\mathcal{L} \to \mathcal{E}\mathrm{Lens}(\mathcal{L})$ defined by:

$$
\begin{array}{lll}
A & \mapsto & \pi_1 : \binom{A \times A^\perp}{A} \\
A \xrightarrow{f} B & \mapsto & \left(f \quad , \quad A \times A^\perp \xleftarrow{\langle \pi_1, Rf \rangle} A \times B^\perp \xrightarrow{f \times 1} B \times B^\perp \right).
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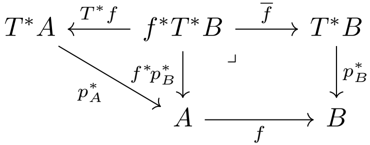
$$A \xrightarrow{f} B \mapsto \left(f \quad , \quad A \times A \xleftarrow{\langle \pi_1, Rf \rangle} A \times B \xrightarrow{f \times 1} B \times B \right)$$

where $Rf : A \times B \to A$ is the reverse differential of f (a primitive data in $\mathcal{L}$).

Differentiation through the lens of lenses

Let $\mathcal{L}$ be a reverse tangent category. This means that $\mathcal{L}$ has a tangent functor T giving tangent bundles $p_A : \binom{TA}{A}$ of objects A and giving tangent arrows $Tf : TA \to TA$ for arrows $f : A \to B$, and we can "reverse" T in order to get cotangent bundles $p_A^* : \binom{T^*A}{A}$ and arrows in the pullback diagram below:



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$$\begin{array}{ccccc} T^*A & \xleftarrow{T^*f} & f^*T^*B & \xrightarrow{\overline{f}} & T^*B \\ & \searrow^{p_A^*} & \downarrow{\scriptstyle f^*p_B^*} & \lrcorner & \downarrow{\scriptstyle p_B^*} \\ & & A & \xrightarrow[f]{} & B \end{array}$$

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We have a functor $\mathcal{L} \to \mathrm{Lens}(\mathcal{L})$ defined by:

$$A \quad \mapsto \quad p_A^* : \binom{T^*A}{A}$$

$$A \xrightarrow{f} B \quad \mapsto \quad \left(\quad f \quad , \quad T^*A \xleftarrow{T^*f} f^*T^*B \xrightarrow{\overline{f}} T^*B \quad \right).$$

Expressing Dialectica as a functor $\Lambda_{\mathrm{cat}} \to \mathcal{E}\mathrm{Lens}(\mathbf{P}_{\mathrm{cat}})$

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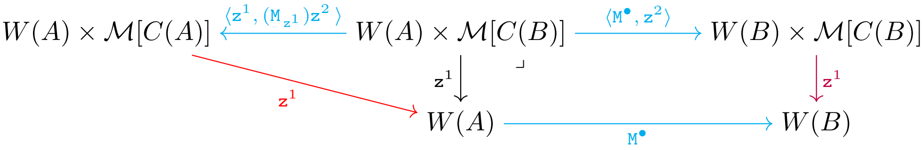
- An object A is sent to the typed term $\mathtt{z} : W(A) \times \mathcal{M}[C(A)] \vdash_{\mathbf{P}} \mathtt{z}^1 : W(A)$
- An arrow $\mathtt{z} : A \vdash_{\Lambda} M : B$ in Λ_{cat} from A to B is sent to the arrow in $\mathcal{E}\text{Lens}(\mathbf{P}_{\text{cat}})$ from $\mathtt{z} : W(A) \times \mathcal{M}[C(A)] \vdash_{\mathbf{P}} \mathtt{z}^1 : W(A)$ to $\mathtt{z} : W(B) \times \mathcal{M}[C(B)] \vdash_{\mathbf{P}} \mathtt{z}^1 : W(B)$ given by the following diagram:



Expressing Dialectica as a functor $\Lambda_{\text{cat}} \to \mathcal{E}\text{Lens}(\mathbf{P}_{\text{cat}})$

Moral:

The Dialectica transformation of λ-calculus encodes (reverse) Differentiation *because* it is a transformation into a category of Lenses, the latter being the abstract setting for Reverse Differentiation.

Final comments

- I didn't talk about Dialectica categories. I could have said something (ask me if you are interested)
- Explore categorical framework to reverse a Cartesian closed differential category in order to define Cartesian closed reverse differential/tangent categories
- Do all with dependent types?
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Grazie, Merci, Thank You!