

Computational interpretations of logic : the case of Dialectica

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	Tarski	Curry-Howard	Kreisel	Krivine	Gödel
<i>extraction language</i>	$\{\square\}$	Type-Theory	$\mathbf{T} + \dots$	$\lambda + \text{callcc} + \dots$	$\mathbf{T} + \dots$
$\mathbb{M} \Vdash A$	$\vDash A$	\mathbb{M} normal $\vdash \mathbb{M} : A$	$\vdash \mathbb{M} : W(A)$ $\vdash \mathbb{M} \text{ mr } A$	$\mathbb{M} \in \text{pl}$ $\forall \rho \in C(A), \mathbb{M} \perp \rho$	$\vdash \mathbb{M} : W(A)$ $\vdash \forall \rho^{C(A)}. \mathbb{M} \perp_A \rho$
$\llbracket \pi \rrbracket =$	\square	$\text{nf}(\tilde{\pi})$	$\tilde{\pi}$	$\tilde{\pi}$	π^+
<i>theorem</i>	<i>sound.</i>	<i>sub. red.</i> <i>+SN+confl</i>	<i>sound.</i>	<i>adequacy</i>	<i>adequacy</i>
\exists proof vs \exists evidence	\iff	\iff	$\not\equiv, \Rightarrow$	$\not\equiv, \Rightarrow$	$\not\equiv, \Rightarrow$

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Let $\llbracket A \rrbracket := \{\mathbf{M} \mid \mathbf{M} \Vdash A\}$.

Question: For a given logic, what is the theory $\{A \mid \llbracket A \rrbracket \neq \emptyset\}$ (and its models)?

Let π be a formal proof of $x : B \vdash A$

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$\exists \text{ proof}$ vs $\exists \text{ evidence}$	\iff	\iff	\neq, \Rightarrow	\neq, \Rightarrow	\neq, \Rightarrow

The jungle of Programs from Proofs

Let π be a formal proof of $x : B \vdash A$

A proof is an algorithm transforming evidence of the hypotheses into evidence of the conclusion:

$$\llbracket \pi \rrbracket : \llbracket B \rrbracket \rightarrow \llbracket A \rrbracket$$

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$[[\pi]] =$	$\square \mapsto \square$	$M \mapsto$ $\text{nf}((\lambda x. \tilde{\pi})M)$	$M \mapsto$ $\tilde{\pi}\{x := M\}$	$M \mapsto$ $\tilde{\pi}\{x := M\}$	$M \mapsto$ $\pi^+\{x := M\}$
<i>theorem</i>	<i>sound.</i>	<i>sub. red.</i> <i>+SN+confl</i>	<i>sound.</i>	<i>adequacy</i>	<i>adequacy</i>
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Gödel
(’38/’58)

$A \in \text{HA}$	\mapsto <i>such that</i>	$w \perp_A c$
$\vdash_{\text{HA}} A$	\implies	<i>there is</i> $\mathbf{M} \in \mathbf{T}$ <i>s.t.</i> $\vdash_{\mathbf{T}} \forall c. \mathbf{M} \perp_A c$

Dialectica in a Nutshell

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Kreisel (\sim ’50)
Kohlenbach (\sim 90)
...

Several variants and extensions (\Rightarrow “Proof Mining”)

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De Paiva (’91)	<i>Dialectica Categories (and models of Linear Logic)</i>

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<p>De Paiva (’91)</p>	<p style="text-align: center;"><i>Dialectica Categories (and models of Linear Logic)</i></p>
<p>Pédrot (’14)</p>	$A \in \text{source} \quad \mapsto \quad W(A), C(A) \in \text{target}$ $\mathbf{M} \in \text{source} \quad \mapsto \quad \mathbf{M}^\bullet, \mathbf{M}_x \in \text{target}$ <p style="text-align: center;"><i>such that</i></p> $\mathbf{x} : B \vdash_{\text{source}} \mathbf{M} : A \quad \implies \begin{cases} \mathbf{x} : W(B) \vdash_{\text{target}} \mathbf{M}^\bullet : W(A) \\ \mathbf{x} : W(B) \vdash_{\text{target}} \mathbf{M}_x : C(A) \rightarrow C(B) \end{cases}$

High-order Weak-Extensional Heyting-Arithmetic (WE-HA^ω)

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- Formulas: $\top, \perp, =_{\text{nat}}, \wedge, \vee, \rightarrow, \forall^X, \exists^X$ and $(t =_{X \rightarrow Y} s) := \forall^X x.(tx =_Y sx)$
 $(A \vee_{b_{\text{nat}}} B) := ((b =_{\text{nat}} 0) \rightarrow A) \wedge ((b \neq_{\text{nat}} 0) \rightarrow B)$

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- Axioms:

$$\begin{aligned}
 & \textit{equality} \\
 & + \\
 & \textit{arithmetic} \\
 & + \\
 & (\text{if } b \text{ then } s \text{ else } t = s) \vee_b (\text{if } b \text{ then } s \text{ else } t = t) \\
 & + \\
 & (\text{rec } f g n = g) \vee_n (\text{rec } f g n = f(n-1)(\text{rec } f g(n-1)))
 \end{aligned}$$

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- Rules:

$$\begin{array}{c}
 \text{Intuitionistic Logic} \\
 + \\
 \frac{A_0 \rightarrow t = s \quad A_0 \text{ quantifier free}}{A_0 \rightarrow B\{x := t\} \rightarrow B\{x := s\}}
 \end{array}$$

Dialectica for $WE-HA^\omega$ in $WE-HA^\omega$

$$\begin{array}{ll} \text{Formulas} & \longrightarrow \text{quant.-free Formulas} \times \mathbf{T} \times \mathbf{T} \\ A & \longmapsto (w \perp_A c, W(A), C(A)) \end{array}$$

Theorem (“Metatheorem” for Dialectica)

Let $\vdash_{WE-HA^\omega} A$. Then

$$\vdash_{WE-HA^\omega} \forall c. \mathbf{a} \perp_A c$$

where $\mathbf{a} \in \mathbf{T}$ is “extracted” from the proof of A

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Theorem (Soundness)

Let $\Delta \vdash_{WE-HA^\omega} A$. Then

$$\Delta' \vdash_{WE-HA^\omega} \forall c. d \perp_{\Delta} c \implies \Delta' \vdash_{WE-HA^\omega} \forall c. a \perp_A c$$

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Theorem (Adequacy)

“Gödel” inductively defines a **program transformation** $(_) \mapsto (_)^+$ which meets the following specification:

If π proves $\mathbf{x} : \Delta \vdash_{WE-HA^\omega} A$, then:

$$\Delta' \vdash_{WE-HA^\omega} \forall c. \mathbf{d} \perp_\Delta c \implies \Delta' \vdash_{WE-HA^\omega} \forall c. \pi^+ \{ \mathbf{x} := \mathbf{d} \} \perp_A c$$

Dialectica for WE-HA^ω in WE-HA^ω

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- If I add the logical principles in Δ , which computational principles should I add in Δ' in order to realise the former?
- Which is the theory Δ of formulas that are realisable (in the empty Δ')?

Dialectica for WE-HA^ω in WE-HA^ω

$$\begin{array}{l} \text{Formulas} \longrightarrow \\ A \longmapsto \end{array} \quad \text{quant.-free Formulas} \times \mathbf{T} \times \mathbf{T} \\ \quad \quad \quad \quad \quad \quad \quad \quad \quad (w \perp_A c \quad , \quad W(A) \quad , \quad C(B))$$

defined by:

$* \perp_{at} *$	$:=$	at	unit	unit
$\langle x, u \rangle \perp_{A \wedge B} \langle y, v \rangle$	$:=$	$(x \perp_A y) \wedge (u \perp_B v)$	$W(A) \times W(B)$	$C(A) \times C(B)$
$\langle b^{\text{nat}}, x, u \rangle \perp_{A \vee B} \langle y, v \rangle$	$:=$	$(x \perp_A y) \vee_{b^{\text{nat}}} (u \perp_B v)$	$\mathbb{N} \times W(A) \times W(B)$	$C(A) \times C(B)$
$\langle f, F \rangle \perp_{A \rightarrow B} \langle x, v \rangle$	$:=$	$x \perp_A Fxv \rightarrow fx \perp_B v$	$\begin{array}{c} W(A) \rightarrow W(B) \\ \times \\ W(A) \rightarrow C(B) \rightarrow C(A) \end{array}$	$W(A) \times C(B)$
$f \perp_{\forall x.A} \langle z, y \rangle$	$:=$	$fz \perp_{A\{x:=z\}} y$	$X \rightarrow W(A)$	$X \times C(A)$
$\langle z, u \rangle \perp_{\exists x.A} y$	$:=$	$u \perp_{A\{x:=z\}} y$	$X \times W(A)$	$C(A)$

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Example

$$\left. \begin{array}{l} \langle f, F \rangle \Vdash A \rightarrow B \\ w \Vdash A \end{array} \right\} \Longrightarrow fw \Vdash B$$

Remember that in Curry-Howard we only have $\text{nf}(fw) \Vdash_{CH} B$.

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We have

$$\vdash \forall x^{W(A)}, v^{C(B)}. x \perp_A Fxv \rightarrow fx \perp_B v \quad \text{and} \quad \vdash \forall y^{C(A)}. w \perp_A y$$

We have to prove that

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Intro c .

Apply the left one with $x := w$, and $v := c$.

Apply the right one with $y := Fxv$.

Apply the left to the right, and we are done.

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$$W(\forall^X x \exists^Y y. \theta(x, y)) = X \rightarrow Y \times \text{unit}$$

$$C(\forall^X x \exists^Y y. \theta(x, y)) = X \times \text{unit}$$

$$\begin{aligned} f \perp_{\forall x \exists y. \theta(x, y)} z, * &= fz \perp_{\exists y. \theta(z, y)} * = (fz)_1, * \perp_{\exists y. \theta(z, y)} * \\ &= * \perp_{\theta(z, (fz)_1)} * \\ &= \theta(z, (fz)_1) \end{aligned}$$

So the Dialectica interpr. of $\forall x \exists y. \theta(x, y)$ is $\sim \forall x. \theta(x, fx)$, for fresh f .

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So the Dialectica interpr. of $\forall x \exists y. \theta(x, y)$ is $\sim \forall x. \theta(x, fx)$, for fresh f .

Corollary

Every provably total function θ in HA is implementable in \mathbf{T} :

$$\vdash_{HA} \forall^X x \exists^Y y. \theta(x, y) \implies \text{there is } f \in \mathbf{T} \text{ such that } \vdash_{WE-HA^\omega} \forall^X x. \theta(x, fx).$$

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Let's compute the Dialectica interpretation of Markov's principle:

$$MP := \neg \forall^X x. \neg \theta(x) \rightarrow \exists^X x. \theta(x)$$

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$$W(MP) = \frac{(\tilde{X} \rightarrow X \times \text{unit})}{(\tilde{X} \rightarrow \text{unit} \rightarrow (X \rightarrow U) \times \text{unit})} \simeq X \rightarrow X$$

$$C(MP) = \tilde{X} \times \text{unit} \simeq X$$

with $U := (\text{unit} \rightarrow \text{unit}) \times (\text{unit} \rightarrow \text{unit} \rightarrow \text{unit}) \simeq \text{unit}$ and

$\tilde{X} := W(\neg \forall^X x. \neg \theta(x)) = ((X \rightarrow U) \rightarrow \text{unit}) \times ((X \rightarrow U) \rightarrow \text{unit} \rightarrow X \times \text{unit} \times \text{unit}) \simeq X$.

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$$\begin{aligned} \langle f, *' \rangle \perp_{MP} \langle *'', \chi, * \rangle &= \neg \neg (\xi_2 \perp_{\theta(\xi_1)} (\star_1 \xi_1)_2 \xi_2 \xi_3) \rightarrow * \perp_{\theta((f \langle *, \chi \rangle)_1)} * \\ &= \neg \neg (* \perp_{\theta(\xi_1)} *) \rightarrow * \perp_{\theta((f \langle *, \chi \rangle)_1)} * \\ &= \neg \neg \theta((\chi \star_1 \star_2)_1) \rightarrow \theta((f \langle *, \chi \rangle)_1) \end{aligned}$$

with $\star = *' \langle *'', \chi \rangle * : (X \rightarrow U) \times \text{unit}$ and $\xi := \chi \star_1 \star_2 : X \times (\text{unit} \times \text{unit})$.

A Few Examples

We can prove (constructively!) $\forall *'', \chi, *. \langle f, *' \rangle \perp_{MP} \langle *'', \chi, * \rangle$ choosing

$$f := \lambda x^{\tilde{X}}. \langle (x_2(*_{X \rightarrow U})*)_1, * \rangle \quad \text{and} \quad *' := \lambda _ , _ . \langle *_{X \rightarrow U}, * \rangle$$

Theorem

*Dialectica realises Markov's Principle: $\Vdash \neg \forall^X x. \neg \theta(x) \rightarrow \exists^X x. \theta(x)$.
Remember that Markov's Principle is not provable constructively!*

$$W(MP) = \frac{(\tilde{X} \rightarrow X \times \text{unit})}{(\tilde{X} \rightarrow \text{unit} \rightarrow (X \rightarrow U) \times \text{unit})} \times \simeq X \rightarrow X$$

$$C(MP) = \tilde{X} \times \text{unit} \simeq X$$

with $U := (\text{unit} \rightarrow \text{unit}) \times (\text{unit} \rightarrow \text{unit} \rightarrow \text{unit}) \simeq \text{unit}$ and

$$\tilde{X} := W(\neg \forall^X x. \neg \theta(x)) = ((X \rightarrow U) \rightarrow \text{unit}) \times ((X \rightarrow U) \rightarrow \text{unit} \rightarrow X \times \text{unit} \times \text{unit}) \simeq X.$$

$$\begin{aligned} \langle f, *' \rangle \perp_{MP} \langle *'', \chi, * \rangle &= \neg \neg (\xi_2 \perp_{\theta(\xi_1)} (*_1 \xi_1)_2 \xi_2 \xi_3) \rightarrow * \perp_{\theta((f\langle *, \chi \rangle)_1)} * \\ &= \neg \neg (* \perp_{\theta(\xi_1)} *) \rightarrow * \perp_{\theta((f\langle *, \chi \rangle)_1)} * \\ &= \neg \neg \theta((\chi *_1 *_2)_1) \rightarrow \theta((f\langle *, \chi \rangle)_1) \end{aligned}$$

with $\star = *' \langle *'', \chi \rangle * : (X \rightarrow U) \times \text{unit}$ and $\xi := \chi *_1 *_2 : X \times (\text{unit} \times \text{unit})$.

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Hoare Triple: $A\langle f\rangle B$

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If the judgment $A\langle f\rangle B$ is derivable, then the formula above is provable (in some ambient theory, say $WE\text{-}HA^\omega$).

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Let’s take this seriously in all its generality:

$$A\langle \mathbf{f} \mid \mathbf{F} \rangle B := \forall s, v. \langle \mathbf{f}, \mathbf{F} \rangle \perp_{A \rightarrow B} \langle s, v \rangle$$

for A, B any formulas. Intuition: $\langle \mathbf{f} \mid \mathbf{F} \rangle$ “realises” $A \rightarrow B$.

Dialectica Hoare Logic (DHL)

Rules for deriving judgments $A \langle \mathbf{f} \mid \mathbf{F} \rangle B$, with $A, B \in \text{WE-HA}^\omega$ and $\mathbf{f}, \mathbf{F} \in \mathbf{T}$

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Usual Soundness Theorem by Gödel. But with the focus on programs \mathbf{f}, \mathbf{F} and DHL as a specification system for them, instead of on formulas.

Dialectica Hoare Logic

$$\perp \langle a \mid - \rangle P \quad P \langle - \mid \alpha \rangle \top \quad P \langle I \mid \text{proj}_2 \rangle P \quad \frac{P_{\exists} \rightarrow Q_{\forall} \in \text{Ax}}{P_{\exists} \langle - \mid - \rangle Q_{\forall}} \quad \frac{P_{\exists} \langle - \mid - \rangle Q_{\forall}}{P'_{\exists} \langle - \mid - \rangle Q'_{\forall}} \text{ for } \frac{P_{\exists} \rightarrow Q_{\forall}}{P'_{\exists} \rightarrow Q'_{\forall}} \in \text{Rule}$$

$$\frac{P \langle a, b \mid \alpha \rangle Q \wedge R}{P \langle b, a \mid \bar{\alpha} \rangle R \wedge Q}^{p\wedge R} \quad \frac{P \wedge Q \langle a \mid \alpha, \beta \rangle R}{Q \wedge P \langle \bar{a} \mid \bar{\beta}, \bar{\alpha} \rangle R}^{p\wedge L} \quad \frac{P \langle a, b \mid \alpha \rangle Q \vee_c R}{P \langle b, a \mid \bar{\alpha} \rangle R \vee_{\bar{c}} Q}^{p\vee R} \quad \frac{P \vee_c Q \langle a \mid \alpha, \beta \rangle R}{Q \vee_{\bar{c}} P \langle \bar{a} \mid \bar{\beta}, \bar{\alpha} \rangle R}^{p\vee L}$$

$$\frac{P \langle a \mid \alpha \rangle Q}{P \langle a, b \mid \alpha_{\pi} \rangle Q \vee_0 R}^{\vee R} \quad \frac{P \langle a \mid \alpha \rangle Q}{P \wedge R \langle a_{\pi} \mid \alpha_{\pi}, \beta \rangle Q}^{\wedge L} \quad \frac{P \langle a, b \mid \alpha \rangle Q \wedge R}{P \langle a \mid \alpha_p \rangle Q}^{\wedge R} \quad \frac{P \vee_0 R \langle a \mid \alpha, \beta \rangle Q}{P \langle a_p \mid \alpha_p \rangle Q}^{\vee L}$$

$$\frac{P \wedge \phi \langle a \mid \alpha \rangle R \quad Q \wedge \neg \phi \langle b \mid \beta \rangle R \quad \phi qf}{P \vee Q \langle \lambda x, y. \text{if } \phi \text{ then } ax \text{ else } by \mid \alpha_{\pi}, \beta_{\pi} \rangle R}^{cond_L} \quad \frac{P \langle a \mid \alpha \rangle Q \quad P \langle b \mid \beta \rangle R}{P \langle a, b \mid \lambda x, v, w. \text{if } P|_{\alpha x v}^x \text{ then } \beta x w \text{ else } \alpha x v \rangle Q \wedge R}^{cond_R}$$

$$\frac{P \langle a, b \mid \alpha \rangle Q \rightarrow R}{P \wedge Q \langle a \mid \alpha, b \rangle R}^{uncurry} \quad \frac{P \wedge Q \langle a \mid \alpha, \beta \rangle R}{P \langle a, \beta \mid \alpha \rangle Q \rightarrow R}^{curry} \quad \frac{P \langle a \mid \alpha \rangle Q \quad Q \langle b \mid \beta \rangle R}{P \langle \lambda x. b(a(x)) \mid \lambda x, w. \alpha x(\beta(ax)w) \rangle R}^{comp}$$

$$\frac{P \langle a \mid \alpha \rangle Q(t)}{P \langle \lambda_{-}. t, a \mid \alpha \rangle \exists x Q(x)}^{\exists R} \quad \frac{P(t) \langle a \mid \alpha \rangle Q}{\forall x P(x) \langle \lambda f. a(ft) \mid \lambda_{-}. t, \lambda f. \alpha(ft) \rangle Q}^{\forall L}$$

$$\frac{P(x) \langle a \mid \alpha \rangle Q}{\exists x P(x) \langle \lambda x. a \mid \lambda x. \alpha \rangle Q}^{\exists L (x \notin Q)} \quad \frac{P \langle a \mid \alpha \rangle Q(x)}{P \langle \lambda y, x. ay \mid \lambda y, x. \alpha y \rangle \forall x Q(x)}^{\forall R (x \notin P)}$$

$$\frac{\exists x P(x) \langle a \mid \alpha \rangle Q}{P(t) \langle at \mid \alpha t \rangle Q}^{sL} \quad \frac{P \langle a \mid \alpha \rangle \forall x Q(x)}{P \langle \lambda y. ayt \mid \lambda y, v. \alpha ytv \rangle Q(t)}^{sR} \quad \frac{P_{\forall} \langle a, b \mid \alpha \rangle \exists x Q(x)}{P_{\forall} \langle b \mid \alpha \rangle Q(a)}^{\epsilon R} \quad \frac{\forall x P_{\forall}(x) \langle - \mid \alpha, \beta \rangle Q_{qf}}{P_{\forall}(\alpha) \langle - \mid \beta \rangle Q_{qf}}^{\epsilon L}$$

$$\frac{P' \langle I \mid \text{proj}_2 \rangle P \quad P \langle a \mid \alpha \rangle Q \quad Q \langle I \mid \text{proj}_2 \rangle Q'}{P' \langle a \mid \alpha \rangle Q'}^{cons} \quad \frac{P \langle a \mid \alpha \rangle Q \quad a, \alpha = b, \beta}{P \langle b \mid \beta \rangle Q}^{ext} \quad \frac{P(x) \langle a(x) \mid \alpha(x) \rangle P(x+1)}{P(0) \langle \mathbf{mc} \ a \mid \mathbf{mc}^* \alpha \rangle \forall x. P(x)}^{-ind}$$

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Update WE-HA^ω

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- Term PL: $\dots \mid \prec: X \rightarrow X \rightarrow \mathbf{nat}$
 $\mid \mathbf{whilerec}_{\phi,a} : (X \rightarrow U) \rightarrow (X \rightarrow U \rightarrow U) \rightarrow X \rightarrow U$

- Formulas: same as before

- Axioms: same as before + the following for $\phi\{x\}$ q.f.:

$$(\phi\{x := y\} \rightarrow ay \prec y) \rightarrow$$

$$\mathbf{whilerec}_{\phi,a} u F y =_U \mathbf{if} \phi\{x := y\} \mathbf{then} F y (\mathbf{whilerec}_{\phi,a} u F (ay)) \mathbf{else} (uy)$$

- Rules: same as before + $\frac{\forall x. ((\forall y \prec x. A\{x := y\}) \rightarrow A)}{\forall x. A}$ (*w.-f. induction*)

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Remark

The sugars

$$\mathbf{while} \phi \mathbf{do} a \quad := \quad \mathbf{whilerec}_{\phi,a} \quad \mathbf{I} \quad \mathbf{proj}_2 \quad : X \rightarrow X$$

$$\mathbf{while}^* \phi \mathbf{do} (a, \alpha) \quad := \quad \mathbf{whilerec}_{\phi,a} \quad \mathbf{proj}_2 \quad (\lambda x, f, v. \alpha x(fv)) \quad : X \rightarrow V \rightarrow V$$

behave in WE-HA^ω like a usual *well-founded* while and a backward while, resp.

Dialectica with While

Add to DHL the rule:

$$\frac{\exists x. (P_{\forall}(x) \wedge \phi(x)) \langle a \mid \alpha \rangle \exists x. P_{\forall}(x) \quad \forall x. (\phi(x) \rightarrow ax \prec x)}{\exists x. P_{\forall}(x) \langle \mathbf{while} \phi \mathbf{do} a \mid \mathbf{while}^* \phi \mathbf{do} (a, \alpha) \rangle \exists x. (P_{\forall}(x) \wedge \neg \phi(x))}$$

Theorem

Dialectica Hoare Logic Soundness keeps holding, i.e. the \mathbf{f}, \mathbf{F} (now with possible loops) in $A \langle \mathbf{f} \mid \mathbf{F} \rangle B$ still realise $A \rightarrow B$ (now with well-founded induction).

A minimum principle:

$$\exists x. \theta \vdash \exists x. (\theta \wedge \forall y \prec x. \neg \theta(y))$$

with \prec well-founded and $\theta\{x^X\}$ quantifier-free.

$$\frac{\frac{\frac{\theta \wedge \phi_g \langle - \mid - \rangle \theta(gx)}{\theta \wedge \phi_g \langle gx \mid - \rangle \exists x. \theta} \exists_R}{\exists x. (\theta \wedge \phi_g) \langle g \mid - \rangle \exists x. \theta} \exists_L \quad \forall x. (\phi_g \rightarrow gx \prec x)}{\frac{\exists x. \theta \langle \mathbf{while} \phi_g \mathbf{do} g \mid - \rangle \exists x. (\theta \wedge \neg \phi_g)}{\exists x. \theta \langle \lambda x, g. (\mathbf{while} \phi_g \mathbf{do} g)x \mid - \rangle \forall g \exists x. (\theta \wedge \neg \phi_g)} \forall_R} \text{while} \quad \exists x. \theta \langle \lambda x, g. (\mathbf{while} \phi_g \mathbf{do} g)x \mid - \rangle \neg \neg \exists x. (\theta \wedge \forall y \prec x. \neg \theta(y)) \quad N$$

with $\phi_g := gx \prec x \wedge \theta(gx)$.

Idea: trial-and-error. (Appears very often in proof mining).

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Fix fresh sets of commands \vec{Comm} , \overleftarrow{Comm} of new types $S \rightarrow S$ and $S \rightarrow T \rightarrow T$.
 $\text{LOOP}_D := \text{IMP}$ with commands from above and *without* variable allocation:

$$C ::= \text{skip} \mid \langle c \mid \gamma \rangle \mid C; C \mid \text{if } \phi \text{ then } C \text{ else } C \mid \text{while } \phi \text{ do } C$$

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Define a translation $\text{LOOP}_D \rightarrow \mathbf{T}^{S \rightarrow S} \times \mathbf{T}^{S \rightarrow T \rightarrow T}$ as:

LOOP_D	$(_)^+$	$(_)^-$
$\langle c \mid \gamma \rangle$	c	γ
skip	I	proj₂
$C_1; C_2$	$\lambda x. C_2^+(C_1^+ x)$	$\lambda x, w. C_1^- x (C_2^- (C_1^+ x) w)$
if ϕ then C_1 else C_2	$\lambda s. \text{if } \phi(s) \text{ then } C_1^+ s \text{ else } C_2^+ s$	$\lambda s, t. \text{if } \phi(s) \text{ then } C_1^- s t \text{ else } C_2^- s t$
while ϕ do C	while ϕ do C^+	(while* ϕ do C^+), C^-

“Hoare Logic” for LOOP_D

$$\frac{}{[P] \text{ skip } [P]} \quad \frac{P(s, \gamma st) \rightarrow Q(cs, t) \in \text{Ax}}{[P] \langle c | \gamma \rangle [Q]} \quad \frac{[P] C_1 [Q] \quad [Q] C_2 [R]}{[P] C_1; C_2 [R]}$$

$$\frac{[P \wedge \phi(s)] C_1 [R] \quad [Q \wedge \neg \phi(s)] C_2 [R]}{[P \vee_{\phi(s)} Q] \text{ if } \phi(s) \text{ then } C_1 \text{ else } C_2 [R]} \quad \frac{[P \wedge \phi(s)] C [P] \quad \phi(s)(s) \rightarrow C^+ s \prec s}{[P] \text{ while } \phi(s) \text{ do } C [P \wedge \neg \phi(s)]}$$

$$\frac{P' \rightarrow P \quad [P] C [Q] \quad Q \rightarrow Q'}{[P'] C [Q']}$$

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Theorem (Soundness wrt Dialectica)

Restrict the above rules to quantifier-free formulas with only one variable s^S and one t^T .

If $[P] C [Q]$ is derivable then DHL derives $\exists^S s \forall^T t. P \langle C^+ | C^- \rangle \exists^S s \forall^T t. Q$ and so

$$\vdash_{\text{WE-HA}^\omega} \forall s, t. (P\{t := C^- st\} \rightarrow Q\{s := C^+ s\})$$

Big-step Operational semantics of LOOP_D

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Forward OS: $\mathbf{T}^S \times \mathbf{LOOP}_D \Downarrow \mathbf{T}^S \times (\mathbf{T}^S)^* \times (\mathbf{T}^{S \rightarrow T \rightarrow T})^*$

$$\frac{}{s, \text{skip} \Downarrow s, [], []} \quad \frac{}{s, \langle c | \gamma \rangle \Downarrow cs, s :: [], \gamma :: []} \quad \frac{s, C_1 \Downarrow s', \sigma, \Gamma \quad s', C_2 \Downarrow s'', \sigma', \Gamma'}{s, C_1; C_2 \Downarrow s'', \sigma' :: \sigma, \Gamma' :: \Gamma}$$

$$\frac{\phi(s) \quad s, C_1 \Downarrow s', \sigma, \Gamma}{s, \text{if } \phi \text{ then } C_1 \text{ else } C_2 \Downarrow s', \sigma, \Gamma} \quad \frac{\neg\phi(s) \quad s, C_2 \Downarrow s', \sigma, \Gamma}{s, \text{if } \phi \text{ then } C_1 \text{ else } C_2 \Downarrow s', \sigma, \Gamma}$$

$$\frac{\neg\phi(s)}{s, \text{while } \phi \text{ do } C \Downarrow s, [], []} \quad \frac{\phi(s) \quad s, C \Downarrow s', \sigma, \Gamma \quad s' \prec s \quad s', \text{while } \phi \text{ do } C \Downarrow s'', \sigma', \Gamma'}{s, \text{while } \phi \text{ do } C \Downarrow s'', \sigma' :: \sigma, \Gamma' :: \Gamma}$$

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Backward OS: $(\mathbf{T}^S)^* \times (\mathbf{T}^{S \rightarrow T \rightarrow T})^* \times \mathbf{T}^T \Downarrow (\mathbf{T}^S)^* \times (\mathbf{T}^{S \rightarrow T \rightarrow T})^* \times \mathbf{T}^T$

$$\frac{}{\sigma, \Gamma, t \Downarrow \sigma, \Gamma, t} \quad \frac{}{s :: \sigma, \gamma :: \Gamma, t \Downarrow \sigma, \Gamma, \gamma st} \quad \frac{\sigma, \Gamma, t \Downarrow \sigma', \Gamma', t' \quad \sigma', \Gamma', t' \Downarrow \sigma'', \Gamma'', t''}{\sigma, \Gamma, t \Downarrow \sigma'', \Gamma'', t''}$$

Big-step Operational semantics of $LOOP_D$

Theorem (Forward+Backward OS = Backpropagation in $LOOP_D$)

Restrict $LOOP_D$ to only terminating (in $WE-HA^\omega$) while loops.

Let C be a command. Then:

For all $s : S$ there exist $\sigma : S^*$ and $\Gamma : (S \rightarrow T \rightarrow T)^*$ such that

①

$$s, C \Downarrow^{\vec{}} (C^+ s), \sigma, \Gamma$$

② for all $t : T$,

$$\sigma, \Gamma, t \Downarrow^{\sim} [], [], (C^- st).$$

Dialectica can be used to implement high-order backpropagation ! (already hinted by Kerjean, Pédrot)

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- The following rule is admissible in DHL:

$$\frac{P_1 \langle a \mid \alpha \rangle Q_1 \quad P_2 \langle b \mid \beta \rangle Q_2}{P_1 \wedge P_2 \langle a, b \mid \alpha, \beta \rangle Q_1 \wedge Q_2}$$

Here, a, α and b, β operate on disjoint variables. So **frame rule**?

$$\frac{P_1 \langle a \mid \alpha \rangle Q_1 \quad P_2 \langle b \mid \beta \rangle Q_2}{P_1 * P_2 \langle a, \alpha \rangle \parallel \langle b, \beta \rangle Q_1 * Q_2}$$

- **Dialectica for Bunched/Separation Logic?**

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- Proof-mining = *quantitative results from qualitative ones*. The algorithmic idea is often by trial-and-error, like our `while ϕ do a` : **make this formal?**

To go further

- Think of S and T as partial $\text{HEAP} \rightarrow \mathbb{N}$ in WE-HA^ω . Then we should be able to have a **variable allocation Dialectica-Hoare rule**?
- The following rule is admissible in DHL:

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Here, a, α and b, β operate on disjoint variables. So **frame rule**?

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Thank you!