Stability Property for the Call-by-Value λ-calculus through Taylor Expansion

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Outline

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1. Call-by-Value
2. Resource approximation for Call-by-Value
3. Stability

Syntax

$$\Lambda \ (Terms) \quad M ::= V \mid MM$$

$$(Values) \quad V ::= x \mid \lambda x.M$$

CbV-operational semantics

Given by a confluent reduction: $(\lambda x.M)N$ is fired only if N is a value.

Example

For $\Delta := \lambda x.xx$, we have:

$\Delta(Ix) \to_{CbV} \Delta x \to_{CbV} xx$ Valid in CbV

$\Delta(Ix) \to_{\beta} (Ix)(Ix) \to_{\beta} x(Ix) \to_{\beta} xx$ Not valid in CbV !

Outline

Syntax

$$\Lambda^r \ \textit{(Resource Terms)} \quad s ::= [v, \ldots, v] \mid ss$$

$$\textit{(Resource Values)} \quad v ::= x \mid \lambda x.s$$

CbV-operational semantics

Given by a confluent and <u>strongly normalising</u> reduction from terms to <u>finite</u> sets of terms.

Example

For $\delta := [\lambda x.[x][x]]$ and $I := [\lambda x.x]$, we have:

$\delta[\,] \to \emptyset$

$\delta[I[x]] \to \delta[x] \to \emptyset$ but no $\delta[I[x]] \to [I[x]][I[x]]$

$\delta[I[x], I[y]] \twoheadrightarrow \delta[x, y] \to \{[x][y], [y][x]\}$ but no $\delta[I[x], I[y]] \to [I[x]][I[y]]$

$\delta[I[x], I[y], I[z]] \twoheadrightarrow \delta[x, y, z] \to \emptyset$ but no $\delta[I[x], I[y], I[z]] \to [I[x]][I[y]]$

It is the map

$$\mathcal{T} : \Lambda \longrightarrow \mathscr{P}(\Lambda^r)$$

defined by induction as:

$$\begin{array}{rcl} \mathcal{T}(x) & := & \{[x, \overset{(n)}{\ldots}, x] \mid n \in \mathbb{N}\} \\ \mathcal{T}(\lambda x.M) & := & \{[\lambda x.s_1, \overset{(n)}{\ldots}, \lambda x.s_n] \mid n \in \mathbb{N},\ s_1, \ldots, s_n \in \mathcal{T}(M)\} \\ \mathcal{T}(MN) & := & \{s_1 s_2 \mid s_1 \in \mathcal{T}(M_1),\ s_2 \in \mathcal{T}(M_2)\} \end{array}$$

Also define

$$\mathrm{NF}\mathcal{T} : \Lambda \longrightarrow \mathscr{P}(\Lambda^r), \qquad \mathrm{NF}\mathcal{T}(M) := \bigcup_{s \in \mathcal{T}(M)} \mathrm{nf}(s)$$

We have an induced partial preorder on Λ:

$$M \leq N \quad \textit{iff} \quad \mathrm{NF}\mathcal{T}(M) \subseteq \mathrm{NF}\mathcal{T}(N)$$

and its induced equivalence.
The quotient $\Lambda/_{\mathrm{NF}\mathcal{T}}$ is partially preordered by $\leq$.

Monotonicity Property

Contexts $C : \Lambda/_{\mathrm{NF}\mathcal{T}} \times \overset{(n)}{\cdots} \times \Lambda/_{\mathrm{NF}\mathcal{T}} \longrightarrow \Lambda/_{\mathrm{NF}\mathcal{T}}$ are always monotone functions

λ-theory

The equivalence $\mathrm{NF}\mathcal{T}$ is a λ-theory

Capturing normal forms

If $s \in \mathrm{NF}\mathcal{T}(M)$ then $\exists\, N$ s.t. $M \twoheadrightarrow N$ and $s \in \mathcal{T}(N)$

Partition Property

$\mathrm{NF}\mathcal{T}(M)$ is partitioned by the family $\{\mathrm{nf}(s) \mid s \in \mathcal{T}(M) \text{ and } \mathrm{nf}(s) \neq \emptyset\}$.

Outline

Theorem (Stability Property)

Let $C : \Lambda/_{\mathrm{NF}\mathcal{T}} \times \overset{(n)}{\cdots} \times \Lambda/_{\mathrm{NF}\mathcal{T}} \longrightarrow \Lambda/_{\mathrm{NF}\mathcal{T}}$ *be a context.*
Let $\mathcal{X}_1, \ldots, \mathcal{X}_n$ *non-empty sets of values bounded in* $\Lambda/_{\mathrm{NF}\mathcal{T}}$ *by a value.*
If all $\inf \mathcal{X}_i$*'s are* definable in $\Lambda/_{\mathrm{NF}\mathcal{T}}$[a] *by a value, then in* $\Lambda/_{\mathrm{NF}\mathcal{T}}$ *we have:*

$$C\langle \inf_{N_1 \in \mathcal{X}_1} N_1, \ldots, \inf_{N_n \in \mathcal{X}_n} N_n \rangle = \inf_{\substack{N_1 \in \mathcal{X}_1 \\ \cdots \\ N_n \in \mathcal{X}_n}} C\langle N_1, \ldots, N_n \rangle.$$

[a] I.e. there is $V \in \Lambda$ s.t. $\mathrm{NF}\mathcal{T}(V) = \bigcap_{N \in \mathcal{X}_i} \mathrm{NF}\mathcal{T}(N)$.

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Corollary (No parallel-or)

There is no term **por** *with the following specification in* $\Lambda/_{\mathrm{NF}\mathcal{T}}$*:*

$$\begin{cases} \mathtt{por}(M, N) = \mathtt{True} & \text{if } M \neq \Omega \text{ or } N \neq \Omega \\ \mathtt{por}(M, N) = \Omega & \text{if } M = \Omega = N \end{cases}$$

Theorem (Stability Property)

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Proof sketch for $n = 1$.

Hypotheses: $\exists\, L$ value s.t. $\mathrm{NF}\mathcal{T}(N) \subseteq \mathrm{NF}\mathcal{T}(L)\ \ \forall N \in \mathcal{X}$

and $\exists\, V$ value s.t. $\mathrm{NF}\mathcal{T}(V) = \bigcap_{N \in \mathcal{X}} \mathrm{NF}\mathcal{T}(N)$.

If suffices to prove that: $\mathrm{NF}\mathcal{T}(C\langle V \rangle) = \bigcap_{N \in \mathcal{X}} \mathrm{NF}\mathcal{T}(C\langle N \rangle)$.

($\subseteq$): immediate by Monotonicity.
($\supseteq$): non-trivial. $\square$

Fix $N \in \mathcal{X}$.

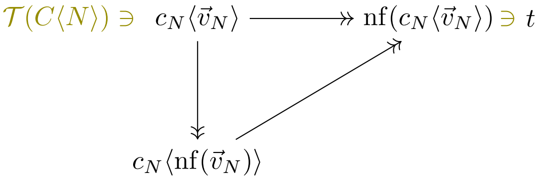
t

Fix $N \in \mathcal{X}$.

$$\mathcal{T}(C\langle N\rangle) \ni \; c_N\langle\vec{v}_N\rangle \longrightarrow\!\!\!\!\!\rightarrow \mathrm{nf}(c_N\langle\vec{v}_N\rangle) \ni t$$

Proof sketch of: $t \in \bigcap_{N \in \mathcal{X}} \mathrm{NF}\mathcal{T}(C\langle N \rangle) \Rightarrow t \in \mathrm{NF}\mathcal{T}(C\langle V \rangle)$

Fix $N \in \mathcal{X}$.



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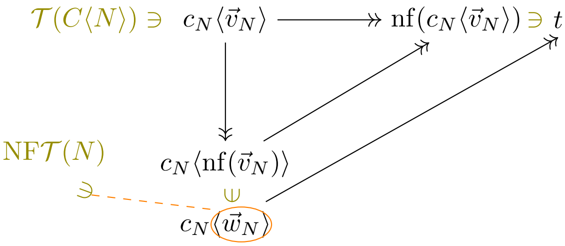

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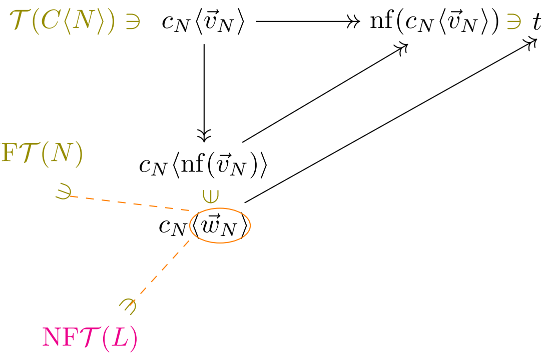

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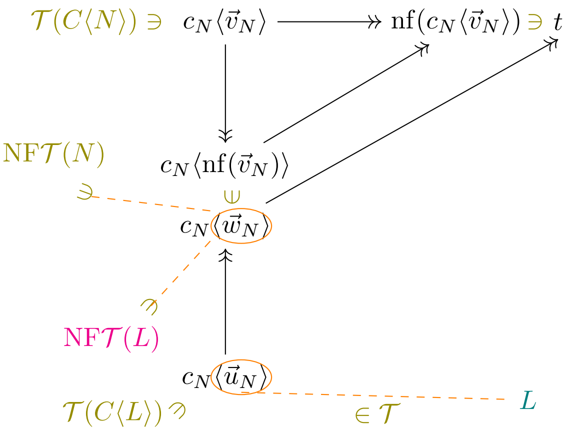

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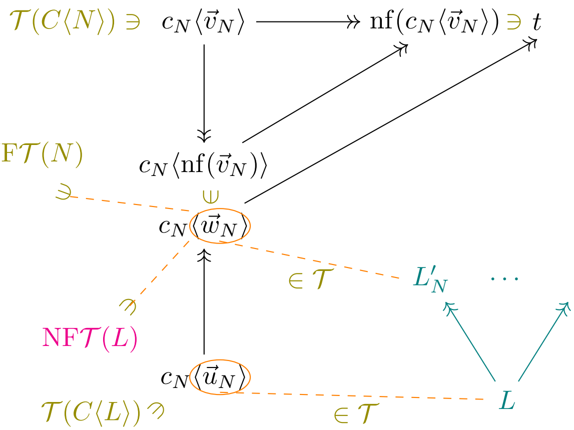
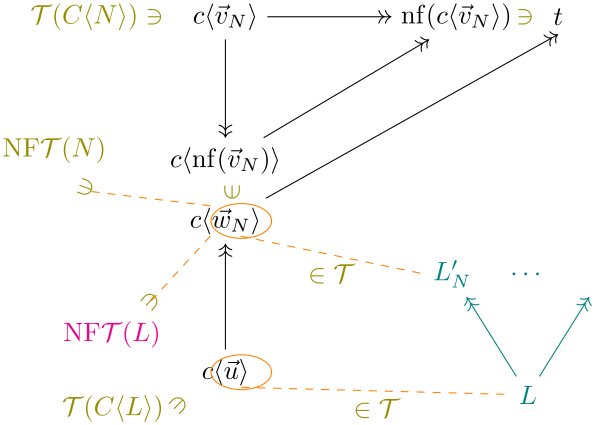


Proof sketch of: $t \in \bigcap_{N\in\mathcal{X}} \mathrm{NF}\mathcal{T}(C\langle N\rangle) \Rightarrow t \in \mathrm{NF}\mathcal{T}(C\langle V\rangle)$



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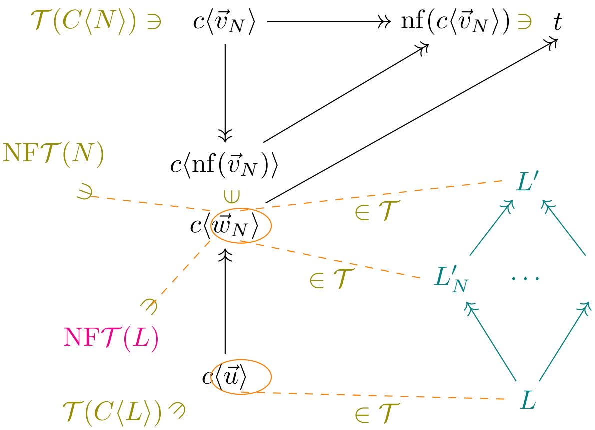

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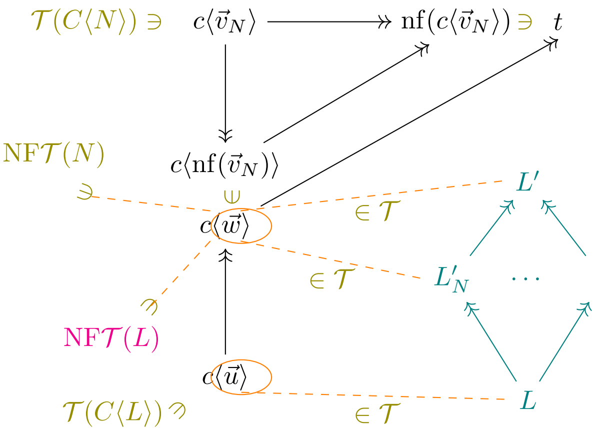

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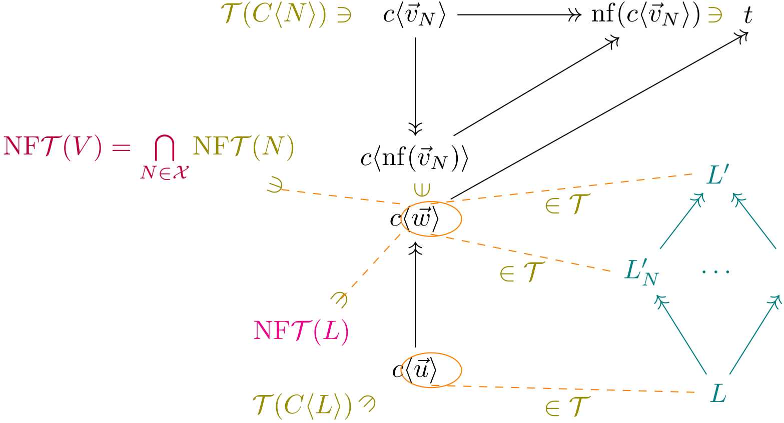

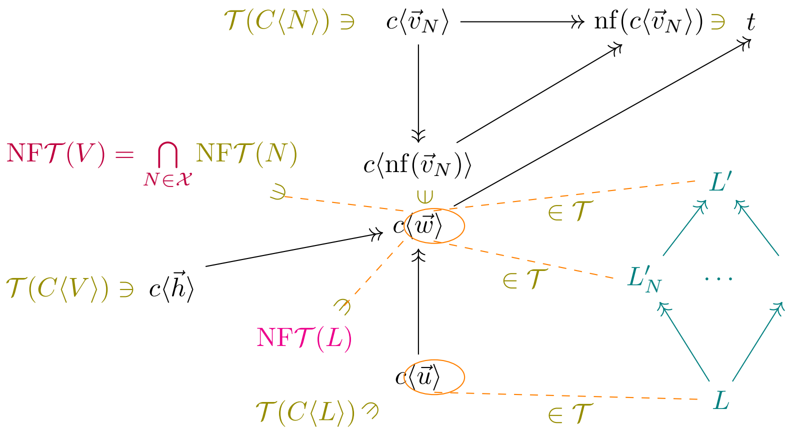
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$\ni$
$\cup$
$c\langle\vec{w}\rangle$
L'
$\in \mathcal{T}$
$\mathcal{T}(C\langle V\rangle) \ni c\langle\vec{h}\rangle$
$\in \mathcal{T}$
L'_N
$\cdots$
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$c\langle\vec{u}\rangle$
L
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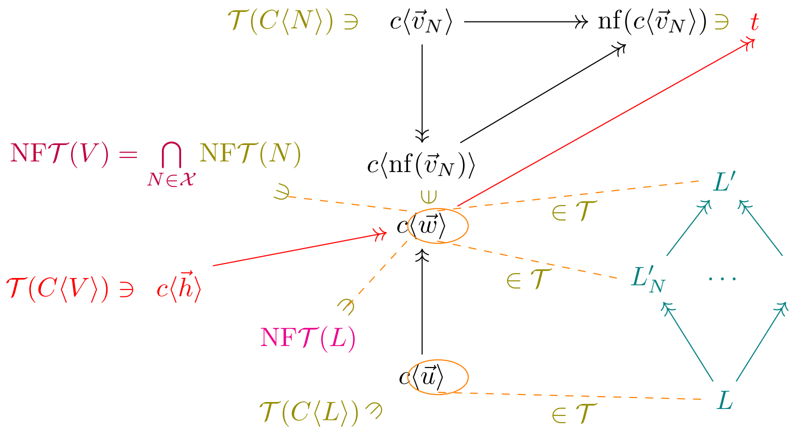
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That's all Folks!