Tropical Mathematics and Linearity in the λ-Calculus

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Linearity

How many duplications/erasures to normal form ?

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Two *orthogonal* approaches:

- *Metric approach*: “Duplication as sensitivity” – easy terms, difficult types – types handle duplication
- *Differential approach*: “Duplication as linear substitution” – difficult terms, easy types – terms handle duplication

Bounded duplication STλC (bSTλC) – syntax

$$\mathtt{M} ::= \mathtt{x} \mid \lambda\mathtt{x}.\mathtt{M} \mid \mathtt{MM} \qquad A ::= X \mid !_n A \multimap A$$

$$\frac{}{\mathtt{x} :_1 A \vdash \mathtt{x} : A} \qquad \frac{\Gamma \vdash \mathtt{M} : A}{\Gamma, \mathtt{x} :_0 B \vdash \mathtt{M} : A} \qquad \frac{\Gamma, \mathtt{x} :_n B, \mathtt{y} :_m B \vdash \mathtt{M} : A}{\Gamma, \mathtt{x} :_{n+m} B \vdash \mathtt{M}\{\mathtt{x}/\mathtt{y}\} : A}$$

$$\frac{\Gamma, \mathtt{x} :_n A \vdash \mathtt{M} : B}{\Gamma \vdash \lambda\mathtt{x}.\mathtt{M} :!_n A \multimap B} \qquad \frac{\Gamma \vdash \mathtt{M} :!_n A \multimap B \qquad \Delta \vdash \mathtt{N} : A}{\Gamma + n\Delta \vdash \mathtt{MN} : B}$$

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$$\mathtt{z} :_2 X \vdash (\lambda\mathtt{xy}.\mathtt{yxx})\,\mathtt{z} : !_1(!_1 X \multimap !_1 X \multimap X) \multimap X$$

Differential STλC (ST$\partial\lambda$C) – syntax

$$\mathtt{M} ::= \mathtt{x} \mid \lambda \mathtt{x}.\mathtt{M} \mid \mathtt{M}\mathbb{T} \mid \mathsf{D}[\mathtt{M},\mathtt{M}] \qquad \mathbb{T} ::= \mathtt{0} \mid \mathtt{M} \mid \mathtt{M} + \mathbb{T} \qquad A ::= X \mid A \to A$$

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$$\mathtt{z} : X \vdash \mathrm{D}^2\left[\lambda\mathtt{x}\mathtt{y}.\mathrm{D}^1\left[\mathrm{D}^1\left[\mathtt{y}, \mathtt{x}^1\right]\mathtt{0}, \mathtt{x}^1\right]\mathtt{0}, \mathtt{z}^2\right]\mathtt{0} : (X \to X \to X) \to X$$

(Denotational) Semantics

$\mathrm{bST}\lambda\mathrm{C}$

Can be interpreted in any SMCC + $\mathbb{N}$-graded linear exponential comonad

Ex: pseudo-Metric spaces & Lipschitz functions

$[\![\mathtt{x} :_n A \vdash_{\mathrm{bST}\lambda\mathrm{C}} \mathtt{M} : B]\!]$ is non-expansive from $!_n[\![A]\!] := ([\![A]\!], n \cdot d_{[\![A]\!]})$ to $[\![B]\!]$.

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ST$\partial\lambda$C

Can be interpreted in any CC$\partial\lambda$C (homsets are commutative monoids endowed with a differential operator)

Ex: Weighted relational semantics, a.k.a. *linear algebra + power series*

For Q a (continuous) semiring, QRel is the opposite category of sets and set-indexed matrices with matrix composition and matrix identity.

The ! is finite multisets, the differential operator $D : Q\mathrm{Rel}_!(X, Y) \to Q\mathrm{Rel}_!(X \times X, Y)$.

Metric vs Differential meet at the tropics

Ugo dal Lago to Paolo & me:

"*Is it possible to take a metric perspective on λ-calculus' Taylor expansion?*"

Logarithmic gap

Lipschitz $n\alpha$ vs Polynomial α^n

Can they coexist ?

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A model of linear-λ-calculus, of $\mathrm{bST}\lambda\mathrm{C}$, of $\mathrm{ST}\lambda\mathrm{C}$ and of $\mathrm{ST}\partial\lambda\mathrm{C}$:

$\mathbb{L}$Rel, i.e. QRel for $Q :=$ the *tropical semiring* $\mathbb{L}$, i.e. $[0, \infty]$ with addition the inf (neutral el. ∞) and multiplication the $+$ (neutral el. 0).

In $\mathbb{L}$ we have $n\alpha = \alpha^n$

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$\Rightarrow$ Let's study what happens inside $\mathbb{L}\mathrm{Rel}_!$

Linear/non-linear functions from linear algebra

A matrix $t \in \mathbb{L}^{X \times Y} = \mathbb{L}(X, Y)$ can be identified as always with a *linear* function $t : \mathbb{L}^X \to \mathbb{L}^Y$.

It is the function associated with the *formal* product (transpose)matrix-variables vector $x = \{x_a\}_{a \in X}$. In the tropical world it is:

$$t(x)_b = \inf_{a \in X} \{x_a + t_{a,b}\}$$

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So a matrix $t \in \mathbb{L}^{!X \times Y} = \mathbb{L}_!(X, Y)$ is a *linear* function $t : \mathbb{L}^{!X} \to \mathbb{L}^Y$. One can express t in base X and see it as a *non-linear* function $t^! : \mathbb{L}^X \to \mathbb{L}^Y$.

It is the function associated with the *formal* power series $t^!(x) \in \mathbb{L}[[\{x_a\}_{a \in X}]]^Y$ in $\#X$ *formal* variables $x = \{x_a\}_{a \in X}$ generated by t. In the tropical world it is a *tropical (formal) Laurent series* (tLs):

$$t^!(x)_b = \inf_{\mu \in !X} \{\mu x + t_{\mu,b}\}$$

Tropics make the metric and differential approach coexist

Endow $\mathbb{L}^X$ with the usual $\|\_\|_\infty$-norm

Theorem (Metric perspective on $\mathrm{ST}\partial\lambda\mathrm{C}$ and Taylor expansion)

1. $[\![\mathtt{x} :_n A \vdash_{\mathrm{bST}\lambda\mathrm{C}} \mathtt{M} : B]\!]^! : \mathbb{L}_{>0}^{[\![A]\!]} \to \mathbb{L}^{[\![B]\!]}$ *is Lipschitz.*
2. $[\![\mathtt{x} : A \vdash_{\mathrm{ST}\lambda\mathrm{C}} \mathtt{M} : B]\!]^! : \mathbb{L}_{>0}^{[\![A]\!]} \to \mathbb{L}^{[\![B]\!]}$ *is locally Lipschitz ($[\![A]\!]$,$[\![B]\!]$ finite)*
3. *The Taylor expansion $\mathcal{T}(\mathtt{M})$ decomposes $[\![\mathtt{x} : A \vdash_{\mathrm{ST}\lambda\mathrm{C}} \mathtt{M} : B]\!]^!$ into an* inf *of Lipschitz maps of higher and higher Lipschitz constant.*

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$$[\![\mathtt{z} :_2 X \vdash_{\mathrm{bST}\lambda\mathrm{C}} (\lambda\mathtt{xy}.\mathtt{yxx})\,\mathtt{z} : !_1(!_1 X \multimap !_1 X \multimap X) \multimap X]\!]^! \text{ is the function:}$$

$f : \mathbb{L} \to \mathbb{L}^{!_1(\{0,1\}\times\{0,1\})}$ *given by* $f(z)_{[(1,1)]} = 2z$, $f(z)_{[(1,0)]} = f(z)_{[(0,1)]} = 1z$, $f(z)_\mu = \infty$ *othw*

$$[\![\mathtt{z} : X \vdash_{\mathrm{ST}\partial\lambda\mathrm{C}} \mathrm{D}^2\left[\lambda\mathtt{xy}.\mathrm{D}^1\left[\mathrm{D}^1\left[\mathtt{y},\mathtt{x}^1\right]0,\mathtt{x}^1\right]0,\mathtt{z}^2\right]0 : (X \to X \to X) \to X]\!]^! \text{ is the function:}$$

$$f : \mathbb{L} \to \mathbb{L}^{!(\mathbb{N}\times\mathbb{N})} \text{ given by } f(z)_{[(1,1)]} = 2z,\ f(z)_\mu = \infty \text{ otherwise}$$

(Taking $[\![X]\!] := \{*\}$ in both)

Lots of Taylors!

Given $\Gamma \vdash_{\mathrm{ST\lambda C}} \mathtt{M} : A$, we may consider:

1. Its λ-calculus Taylor expansion $\mathcal{T}(\Gamma \vdash_{\mathrm{ST\lambda C}} \mathtt{M} : A)$
2. The CC∂C Taylor expansion of its interpretation $[\![\Gamma \vdash_{\mathrm{ST\lambda C}} \mathtt{M} : A]\!]$
3. The tropical Taylor expansion of the *formal tLS* $[\![\Gamma \vdash_{\mathrm{ST\lambda C}} \mathtt{M} : A]\!]^!$
4. The math. analysis Taylor expansion of the *function* $[\![\Gamma \vdash_{\mathrm{ST\lambda C}} \mathtt{M} : A]\!]^!$

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1 coincides with 2 in the $\mathbb{L}\mathrm{Rel}_{!}$, and are related to 3.
All three talk about the program.

4 is *a priori* unrelated with the program[1] (it is there because we can see a formal series as a function).

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So what?

Math:

- Study tLs' in general (e.g. get rid of the "$[\![A]\!]$,$[\![B]\!]$ finite" condition)
- The $\mathbb{L}$Rel differential $D : \mathbb{L}^{!X\times Y} \to \mathbb{L}^{!(X+X)\times Y}$ translates into $D_! :$ $\{f : \mathbb{L}^X \to \mathbb{L}^Y \mid f \text{ tLs}\} \to \{f : \mathbb{L}^X \times \mathbb{L}^X \to \mathbb{L}^Y \mid f \text{ linear in its } 1st \text{ var}\}$. Relations with the usual tropical derivative of tropical polynomials
- Differentials of maps between modules/generalised metric spaces

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CS:

- Probabilistic lang.: $[\![\Gamma \vdash_{p\mathbb{P}PCF} \mathtt{M} : A]\!]^{\mathbb{L}\mathrm{Rel}}$ *vs* $[\![\Gamma \vdash_{p\mathbb{P}PCF} \mathtt{M} : A]\!]^{PCoh}$. $[\![\_]\!]^{\mathbb{L}\mathrm{Rel}}$ gives the *tropicalisation of the probability* of any of the *most likely* reduction paths to normal form of the stochastic process
- Differential privacy
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... more to come on that !


That's all Folks!