Denotational semantics driven homology ?

Davide Barbarossa

davide.barbarossa@unibo.it

https://lipn.univ-paris13.fr/~barbarossa/

Dipartimento di Informatica

ALMA MATER STUDIORUM
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Trends in Linear Logic and Applications

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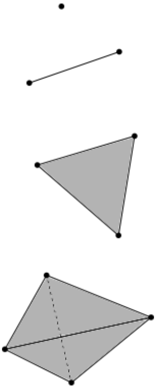
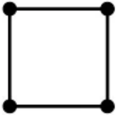
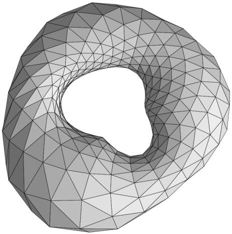
Simplicial complexes

Δ^n | $S^{n-1} = \partial\Delta^n$ | Coherent spaces | Triangulated space







Simplicial homology, in 1 slide!

$$\mathrm{ASC} \xrightarrow{\quad \mathcal{C} \quad} \mathrm{Chain}_{\mathbb{Z}} \xrightarrow{\quad \mathcal{H}_k \quad} \mathrm{Modules}_{\mathbb{Z}}$$

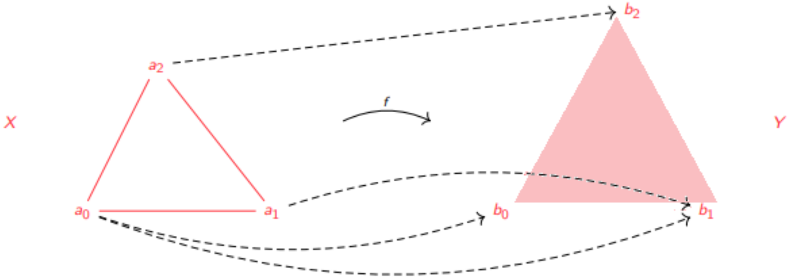
$$X \longmapsto (\mathcal{C}_k X, \partial_k^X)_{k\in\mathbb{N}} \longmapsto \mathcal{H}_k X$$

If $X \xrightarrow{f} Y$ in ASC, then for $\mathcal{C}f$ to be a morphism in $\mathrm{Chain}_{\mathbb{Z}}$ means that:

$$\begin{array}{ccccccc}
\longrightarrow & \mathcal{C}_{k+1}X & \xrightarrow{\partial_{k+1}^X} & \mathcal{C}_k X & \longrightarrow \\
 & \downarrow{\scriptstyle \mathcal{C}_{k+1}f} & & \downarrow{\scriptstyle \mathcal{C}_k f} & \\
\longrightarrow & \mathcal{C}_{k+1}Y & \xrightarrow[\partial_{k+1}^Y]{} & \mathcal{C}_k Y & \longrightarrow
\end{array}$$

In topology, morphisms in ASC are (simplicial) *functions*. But a proof is interpreted as a (simplicial) *relation*! Let us call this category RelASC.

Not clear how to lift $\mathcal{C}$ to a functor $\mathrm{RelASC} \to \mathrm{Chain}_{\mathbb{Z}}$



$$
\begin{array}{ccc}
(a_0, a_2) & \longmapsto & (a_2) - (a_0) \\
\Big\downarrow & & \Big\downarrow \\
(b_0, b_2) + (b_1, b_2) & \longmapsto & 2(b_2) - (b_0) - (b_1) \neq (b_2) - (b_0) - (b_1)
\end{array}
$$

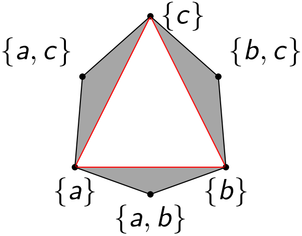
$$
\begin{array}{ccc}
\mathcal{C}_{k+1} X & \xrightarrow{\partial^X_{k+1}} & \mathcal{C}_k X \\
\Big\downarrow {\scriptstyle \mathcal{C}_{k+1} f} & & \Big\downarrow {\scriptstyle \mathcal{C}_k f} \\
\mathcal{C}_{k+1} Y & \xrightarrow[\partial^Y_{k+1}]{} & \mathcal{C}_k Y
\end{array}
$$

We can be functorial! ...but we transform the spaces

We can define a monad $\mathscr{I}$ on ASC such that:

$$\begin{array}{ccccccccc} \mathrm{RelASC} & \cong & \mathrm{ASC}_{\mathscr{I}} & \xrightarrow{R_{\mathscr{I}}} & \mathrm{ASC} & \xrightarrow{\mathcal{C}} & \mathrm{Chain}_{\mathbb{Z}} & \xrightarrow{\mathcal{H}_k} & \mathrm{Modules}_{\mathbb{Z}} \\ X & \longmapsto & X & \longmapsto & \mathscr{I}X & & \longmapsto & & \mathcal{H}_k(\mathscr{I}X) \end{array}$$

An example: S^1 in red and $\mathscr{I}S^1$ in grey. In this case, $\mathcal{H}_kX = \mathcal{H}_k(\mathscr{I}X)$.



And so what?

Fix a "webbed" semantics $[\![.]\!]$. Call $[A]^{[\![\cdot]\!]}$ the asc with vertices $|[\![A]\!]|$ and simplices the $x \subseteq [\![\pi]\!]$, for $\pi :\vdash A$.

Corollary

If A and B are "type-isomorphic", then $\mathcal{H}_k(\mathscr{I}[A]^{Rel}) = \mathcal{H}_k(\mathscr{I}[B]^{Rel})$.

- Morally, $[A]^{Rel}$ represents the *geometrical* realisation of the *space* of the proofs of A, under the relational semantcs. Study its geometry !
- Is $\mathcal{H}_k X = \mathcal{H}_k(\mathscr{I}X)$ true for any X ? If yes, that is nice. If not, give a counterexample.
- Does $\mathscr{I}$ have a logical/computational/geometrical meaning ?
- What about $[A]^{\mathrm{Coh}}$?
- Are n-holes related with sequentiality ? (Think of $[\mathrm{Gustave}]^{Coh}$)


That's all Folks!