An overview of (Tropical) Quantitative Semantics and Taylor Expansion for the λ-calculus

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Outline

1. (Quantitative) Semantics of Programs
2. Quantitative Semantics: Linearity
3. Tropical Polynomials and Effectful Computation
4. Tropically Weighted Relational Semantics of $\mathbb{P}$PCF
5. Overview of our recent results (CSL24 – Barbarossa, Pistone)
6. Future Work
7. Bonus: Finitness, Taylor, Generalised Metric Spaces

Outline

Program semantics $\Rightarrow$ properties of programs

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Qualitative:

- termination
- correctness
- program equivalence
- ...

Program semantics $\Rightarrow$ properties of programs

Qualitative:

- termination
- correctness
- program equivalence
- ...

Quantitative:

- probability of convergence
- probability of correctness
- errors, program similarity
- ...

Sensitivity Analysis

program metrics, quantitative equational theories

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Resource Analysis

linear logic, program differentiation, intersection types

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type $\mapsto$ metric space

program $\mapsto$ Lipschitz function

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program ↦ Lipschitz function

$$M \overset{\epsilon}{\simeq} N \Rightarrow FM \overset{L\cdot\epsilon}{\simeq} FN$$

Metric Preservation

Sensitivity Analysis

program metrics, quantitative equational theories

type	$\mapsto$	metric space
program	$\mapsto$	Lipschitz function

$$M \overset{\epsilon}{\simeq} N \Rightarrow FM \overset{L\cdot\epsilon}{\simeq} FN$$

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type	$\mapsto$	vector space /module
program	$\mapsto$	smooth/analytic function

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$$FM = \sum_{n=0}^{\infty} \frac{1}{!n} \mathsf{D}^n[F, M^n](0)$$

Taylor Formula

λ-calculus (Purely functional, Turing–complete)

$$M ::= x \mid \lambda x.N \mid MN \qquad\qquad (\lambda x.M)N \to M\{x := N\}$$

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Simple types (Lose Turing-completeness, can be recovered easily)

$$\frac{}{x : A \vdash x : A} \qquad \frac{x : A \vdash M : B}{\vdash \lambda x.M : A \to B} \qquad \frac{\vdash M : A \to B \quad \vdash N : A}{\vdash MN : B}$$

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Categorical semantics in a Cartesian Closed Category $\mathcal{C}$

type A	$\mapsto$	object $[\![A]\!]$ in $\mathcal{C}$
program $x : A \vdash M : B$	$\mapsto$	morphism $[\![x : A \vdash M : B]\!] : [\![A]\!] \to [\![B]\!]$ in $\mathcal{C}$

Outline

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Resource Analysis

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	$M \overset{\epsilon}{\simeq} N \Rightarrow FM \overset{\epsilon}{\simeq} FN$	$FM \;=\; (\mathsf{D}F \cdot M)0$

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F uses input once	f non-expansive $M \overset{\epsilon}{\simeq} N \Rightarrow FM \overset{\epsilon}{\simeq} FN$	f linear $(\mathsf{D}^2 F = 0)$ $FM \;=\; (\mathsf{D}F \cdot M)0$
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	graded types $!_k A \multimap B$	intersection types $[A_1, \ldots, A_k] \multimap B$

Graded typed calculus

$$\frac{}{x :!_1 A \vdash x : A}$$

$$\frac{x :!_s C \vdash M :!_n A \multimap B \quad x :!_m C \vdash N : A}{x :!_{s+nm} C \vdash MN : B}$$

$$\frac{x :!_n A \vdash M : B}{\vdash \lambda x.M :!_n A \multimap B}$$

Resource calculus

$$\frac{}{x : [A] \vdash x : A}$$

$$\frac{x : [A_1, \cdots A_n] \vdash M : B}{\vdash \lambda x.M : [A_1, \cdots A_n] \multimap B}$$

$$\frac{\vdash M : [A_1, \cdots A_n] \multimap B \quad (\vdash N_i : A_i)_{i=1}^{n}}{\vdash M[N_1, \cdots N_n] : B}$$

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Taylor expansion of a λ-term

$$\mathcal{T}(x) := \{x\}$$

$$\mathcal{T}(\lambda x.M) := \{\lambda x.t \mid t \in \mathcal{T}(M)\}$$

$$\mathcal{T}(MN) := \{\, t\,[u_1, \ldots, u_n] \mid n \in \mathbb{N},\ t \in \mathcal{T}(M),\ u_1, \ldots, u_n \in \mathcal{T}(N)\}$$

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k times


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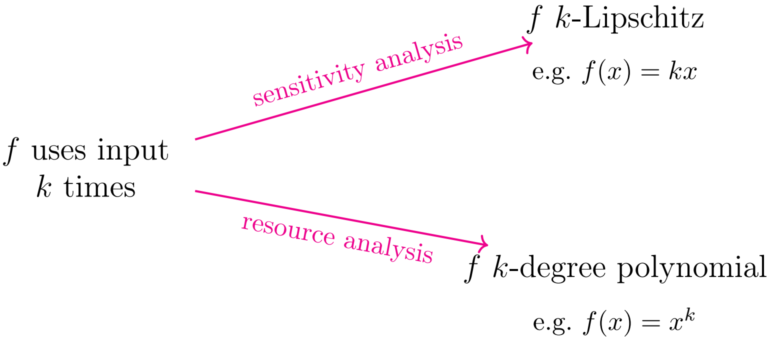
sensitivity analysis
f k-Lipschitz
e.g. $f(x) = kx$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f k-degree polynomial
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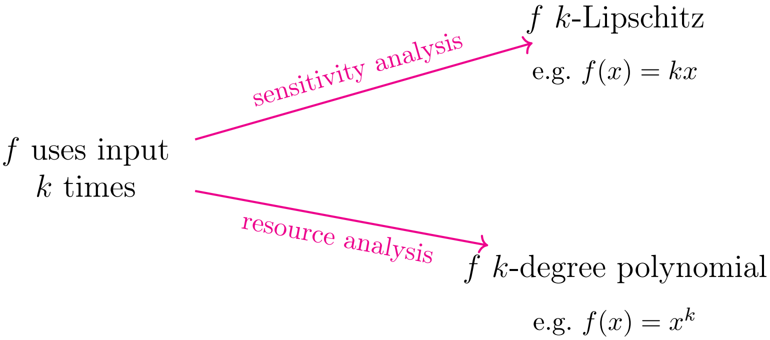
Quantitative Semantics: a Logarithmic Gap



Tropical Mathematics

a k-degree polynomial is a k-Lipschitz function!

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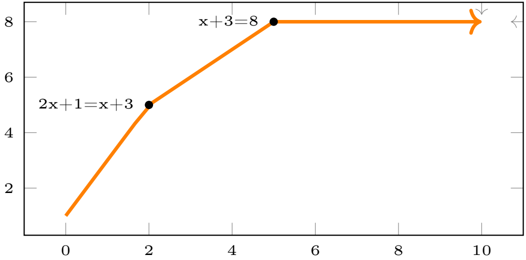

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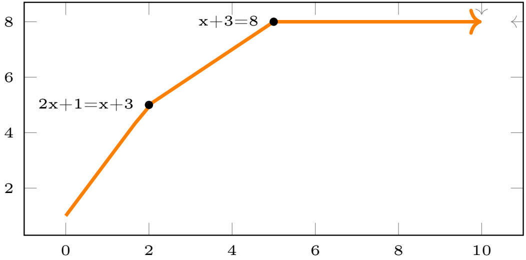
x_0 is a **tropical root** of $p(x)$ iff $p(x_0)$ is not differentiable



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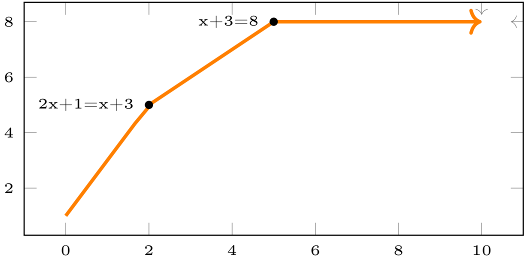
x_0 is a **tropical root** of $p(x)$ iff $p(x_0)$ is not differentiable

equivalently, iff the minimum $p(x_0)$ is attained twice



Intractable problems (e.g. root finding, optimization)

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$\downarrow$ tropicalization:

$+ \mapsto \min$

$\times \mapsto +$

Combinatorial (and sometimes tractable!) ones

- tropical roots are found in linear time
- likelihood estimation in statistical models
- machine learning (ReLU networks)
- optimal routing paths

f uses input
k times

sensitivity analysis →

f k-Lipschitz

e.g. $f(x) = kx$

resource analysis →

f k-degree polynomial

e.g. $f(x) = x^k$

Tropical Mathematics

a k-degree polynomial is a k-Lipschitz function!

f uses input k times

sensitivity analysis → f k-Lipschitz

e.g. $f(x) = kx$

resource analysis → tropical f k-degree polynomial

e.g. $f(x) = kx$

Tropical Mathematics

a k-degree polynomial is a k-Lipschitz function!

Outline

$$M ::= \text{True} \mid \text{False} \mid M \oplus_p M \quad (p \in [0,1] \cap \mathbb{Q})$$

$$M \oplus_p N \twoheadrightarrow_p M$$
$$M \oplus_p N \twoheadrightarrow_{1-p} N$$

$$(\text{True} \oplus_p \text{False}) \oplus_p \Big((\text{True} \oplus_p \text{False}) \oplus_p (\text{False} \oplus_p \text{True}) \Big)$$

$$\left(\mathrm{True} \oplus_p \mathrm{False}\right) \oplus_p \left(\left(\mathrm{True} \oplus_p \mathrm{False}\right) \oplus_p \left(\mathrm{False} \oplus_p \mathrm{True}\right)\right)$$

$$q := 1 - p$$

$$\begin{aligned} P_{ll}(p,q) &= p^2 \\ P_{rll}(p,q) &= p^2 q \\ P_{rrr}(p,q) &= q^3 \end{aligned}$$

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$$P_{\text{True}}(p,q) = p^2 + p^2q + q^3$$

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Hidden Markov Model

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Maximum Likelihood problem:
supposing $M \twoheadrightarrow \text{True}$,
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$$P_{\omega_0}(p,q) = \max_\omega P_\omega(p,q)$$

$$(\text{True} \oplus_p \text{False}) \oplus_p \Big((\text{True} \oplus_p \text{False}) \oplus_p (\text{False} \oplus_p \text{True})\Big)$$

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Maximum Likelihood problem: supposing $M \twoheadrightarrow$ True, what is the most likely path?

→ find $\omega_0 \in \{ll, rll, rrr\}$ minimizing $-\log\, P(M \twoheadrightarrow_{\omega_0} \text{True} \mid M \twoheadrightarrow \text{True})$:

$$-\log\, P_{\omega_0}(p,q) = \min_{\omega}\{-\log\, P_{\omega}(p,q)\}$$

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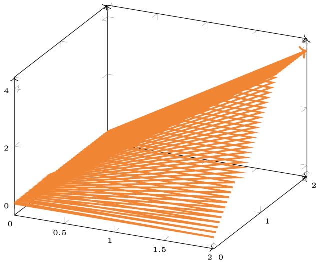

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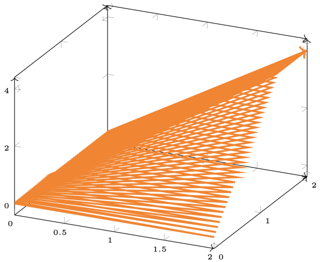
tropical roots $\mapsto$ line $y = \frac{2}{3}x$



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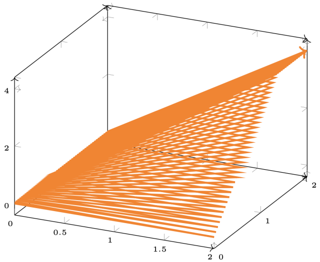

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tropical roots $\mapsto$ line $y = \frac{2}{3}x$

- rrr most likely as soon $1-p \geq p^{\frac{2}{3}}$ (e.g $p = \frac{1}{4}$)
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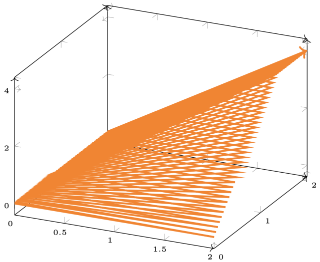

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$$\begin{array}{ll} M \twoheadrightarrow_p \text{True} & p \\ M \twoheadrightarrow_q M \twoheadrightarrow_p \text{True} & qp \\ M \twoheadrightarrow_q M \twoheadrightarrow_q M \twoheadrightarrow_p \text{True} & q^2 p \\ \quad \ldots & \end{array}$$

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$M \twoheadrightarrow_p \mathrm{True}$	p
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Outline

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type A	$\mapsto$	$\mathbb{L}$-module $\mathbb{L}^{[\![A]\!]}$ with metric d_∞
program $x : A \vdash M : B$	$\mapsto$	tropical power series $[\![x : A \vdash M : B]\!] : \mathbb{L}^{[\![A]\!]} \to \mathbb{L}^{[\![B]\!]}$

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F is the limit of more and more sensitive approximations

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Theorem. [$\mathbb{L}$Mod $\simeq$ $\mathbb{L}$CCat is a model of ST$\partial\lambda$C] The equivalent categories of $\mathbb{L}$-modules and complete generalized metric spaces form a model of ST$\partial\lambda$C which extends the $\mathbb{L}$-weighted relational model.

Outline

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Thank you!

Outline

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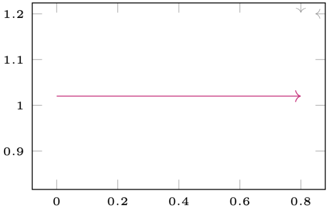

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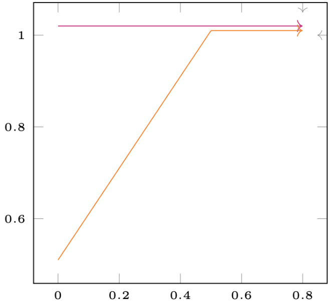

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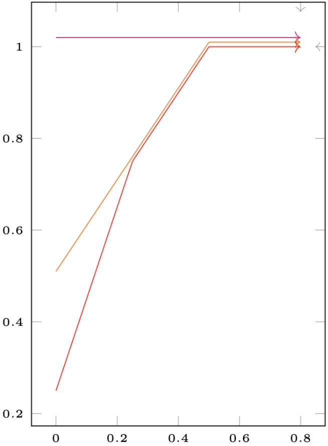

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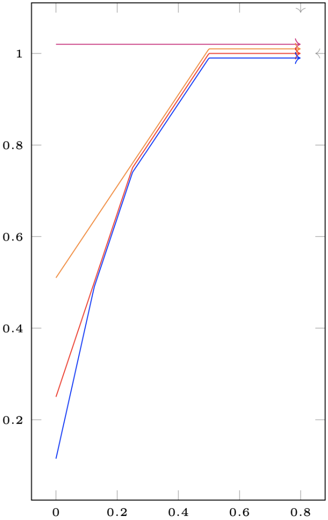

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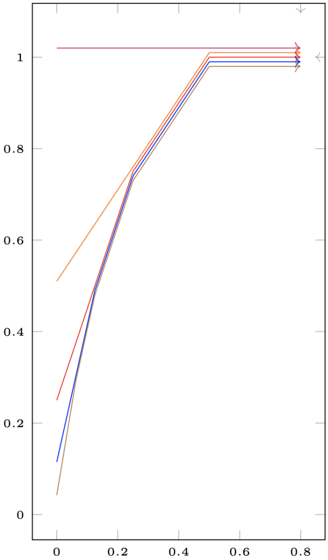

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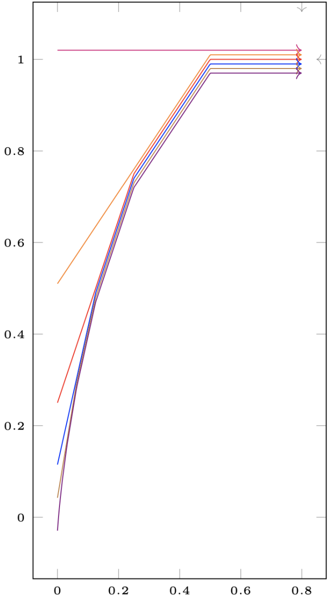

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φ is "locally" a polynomial:

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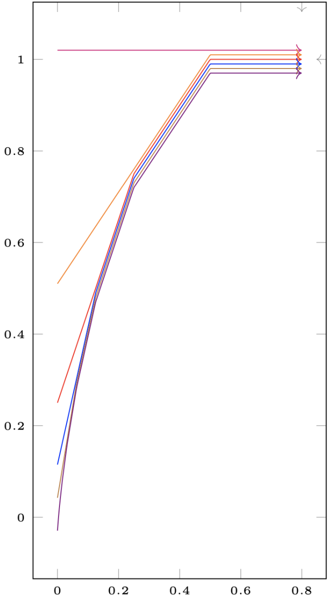

Theorem. Let $f : [0, +\infty]^k \to [0, +\infty]$ be a tropical power series given by

$$f(x) = \inf_{i \in I} \{n_i x + c_i\}.$$

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$\to$ if we can compute the polynomial $[\![M]\!]_n|_{[\epsilon,+\infty]}$ for ϵ small enough, then we can compute maximum likelihood values for M.

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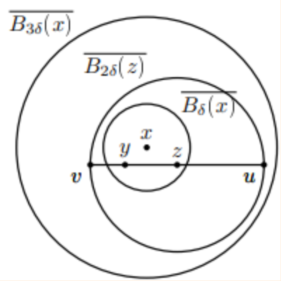

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Corollary. Let $M : A \to B$ and $N : A$. For all $t \in \mathcal{T}(M)$ and $\delta > 0$, unless $[\![t]\!]([\![N]\!]) \neq \infty$, the map $[\![M]\!](x)$ is $\frac{[\![t]\!]([\![N]\!]+3\delta)}{\delta}$-Lipschitz over the open ball $B_\delta([\![N]\!])$.

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- objects are complete generalized metric spaces $(X, a : X \times X \to \mathbb{L})$ (a.k.a. $\mathbb{L}$-enriched categories)

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Theorem. $\mathbb{L}\mathsf{Mod}_! \simeq \mathbb{L}\mathsf{CCat}_!$ extends $\mathbb{L}\mathsf{Rel}_!$ as a model of the ST$\partial\lambda$C