The resource $\lambda\mu$-calculus, and applications

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Curry-Howard correspondence

The intuitionistic case:

Intuitionistic proofs express their computational content inside λ-calculus:

$$M ::= x \mid \lambda x.M \mid MM \qquad (\lambda x.M)N \to M\{N/x\}$$

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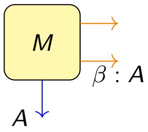
The classical case: (notable cases)

- Krivine's classical realizability: a beautiful "machine" for extracting computational content from proofs + axioms
- $\lambda\mu$-calculus: classical proofs express their computational content in it

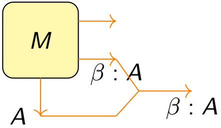
$$M ::= x \mid \lambda x.M \mid MM \mid \mu\alpha._{\beta}|M|$$

The intuition behind ${}_{\beta}|\_|$ and $\mu\beta.\_$



M

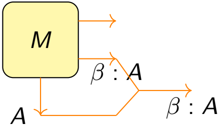
The intuition behind $_{\beta}|\_|$ and $\mu\beta.\_$



$_{\beta}|M|$

Hide information...

The intuition behind ${}_{\beta}|\_|$ and $\mu\beta.\_$



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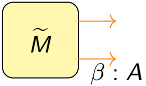

$\widetilde{M}$

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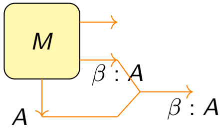

${}_{\beta}|M|$

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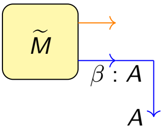
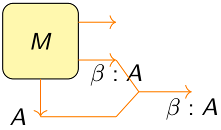


$\mu\beta.\widetilde{M}$

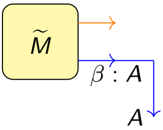
...and retrieve it

The intuition (Translates into Polarized Proof-Nets...)



$_{\beta}|M|$

Hide information...



$\mu\beta.\widetilde{M}$

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The $\lambda\mu$-calculus (Parigot '92)

Terms

$$M ::= x \mid \lambda x.M \mid MM \mid \mu\alpha.{}_{\beta}|M|$$

Reduction

$$(\lambda x.M)N \to_{\lambda} M\{N/x\}$$

$$\mu\alpha.{}_{\beta}|\mu\gamma.{}_{\eta}|M|| \to_{\rho} \mu\alpha.{}_{\eta}|M|\{\beta/\gamma\}$$

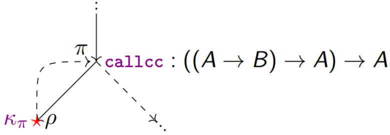
$$(\mu\alpha.{}_{\beta}|M|)N \to_{\mu} \mu\alpha.({}_{\beta}|M|)_{\alpha}N$$

where $({}_{\beta}|M|)_{\alpha}N := {}_{\beta}|M|\{{}_{\alpha}|(\cdot)N|/{}_{\alpha}|\cdot|\}$.

It is an impure functional Prog Lang:

Continuations in $\lambda\mu$-calculus

$$\texttt{callcc} := \lambda y.\mu\alpha.{}_{\alpha}|y(\lambda x.\mu\delta.{}_{\alpha}|x|)|$$



The $\lambda\mu$-calculus (Parigot '92)

Terms

$M ::= x \mid \lambda x.M \mid MM \mid \mu\alpha._\beta|M|$

Reduction

$(\lambda x.M)N \to_\lambda M\{N/x\}$

$M|\{\beta/\gamma\}$

$|)_\alpha N$

where $(_\beta|$

Models of the simply typed λ-calculus, of the untyped λ-calculus and of the simply typed $\lambda\mu$-calculus are well understood, but what about models of the untyped $\lambda\mu$-calculus? As far as we know, this question has been almost ignored.

O. Laurent ('04)

On the denotational semantics of the untyped $\lambda\mu$-calculus

It is an i

Continuations in $\lambda\mu$-calculus

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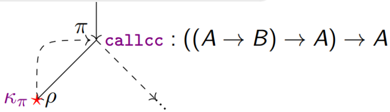
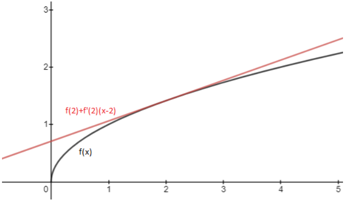


Differential λ-calculus: Taylor expansion $\Theta(F)$ of F



Analysis	λ-calculus
$\sum_n \frac{1}{n!} F^{(n)}(0) x^n$	$\sum_n \frac{1}{n!} (\mathtt{D}^n \Theta(F) \bullet x^n) 0$

Ehrhard and Régnier ('08):

One can define $\Theta : \Lambda \to \mathbb{Q}^+ \langle \Lambda^r \rangle_\infty$ as:

$$\Theta(F) = \sum_{t \in \mathcal{T}(F)} \frac{1}{\mathrm{m}(t)} t$$

where $\mathrm{m}(t) \in \mathbb{N}$ is difficult and $\mathcal{T} : \Lambda \to \mathcal{P}(\Lambda^r)$ is easy (i.e. inductive).

The resource λ-calculus

Define the set Λ^r of Resource terms:

$$t ::= x \mid \lambda x.t \mid t\,[t, \ldots, t]$$

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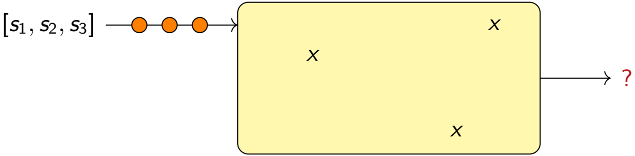
$$(\lambda x.t)[s_1, s_2, s_3] \to ?$$



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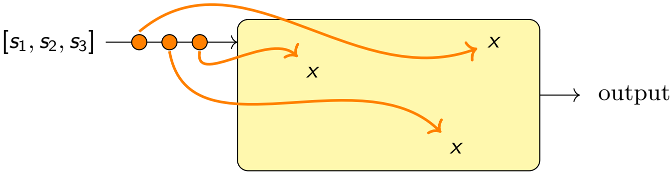
$$(\lambda x.t)[s_1, s_2, s_3] \to t\{s_1/x^{(1)}, s_2/x^{(2)}, s_3/x^{(3)}\}$$



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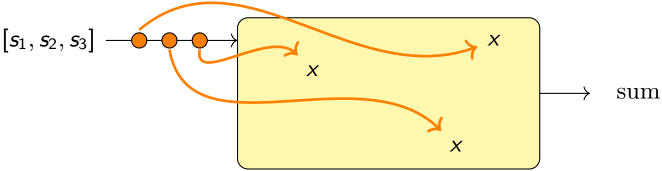
Define the set Λ^r of Resource terms:

$$t ::= \ x \ \mid \lambda x.t \ \mid t\,[t, \dots, t]$$

We need formal (*idempotent*) sum $\mathbb{T} = \ t_1 + \cdots + t_n$ of resource terms.
Reduction:

$$(\lambda x.t)[s_1, s_2, s_3] \to \sum_{\sigma \in \mathfrak{S}_3} t\{s_{\sigma(1)}/x^{(1)}, s_{\sigma(2)}/x^{(2)}, s_{\sigma(3)}/x^{(3)}\}$$



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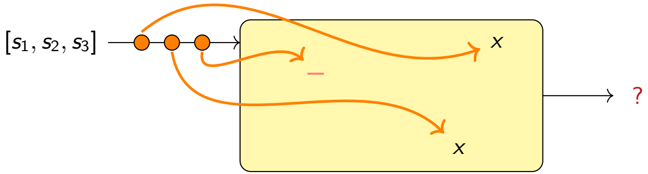

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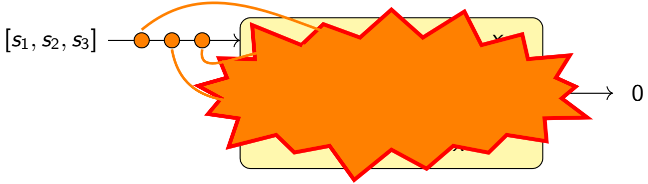

Resource terms live a tough life

They experience:

- Non-determinism: $\Delta[x, y] := (\lambda x.x[x])[y, y'] \to y[y'] + y'[y]$
- Starvation: $\Delta[\Delta, \Delta] \to (\lambda x.x[x])[\Delta] \to 0$
- Surfeit: $(\lambda x \lambda y.x)[I][I] \to (\lambda y.I)[I] \to 0$
- Strong normalization
- Confluence

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- Confl

Understanding the relation between the term and its full Taylor expansion might be the starting point of a renewing of the theory of approximations.
T. Ehrhard, L. Regnier ('03)
The differential lambda-calculus

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They experience:

- Non-determinism ... $[y'] + y'[y]$
- Starv... $A[A$
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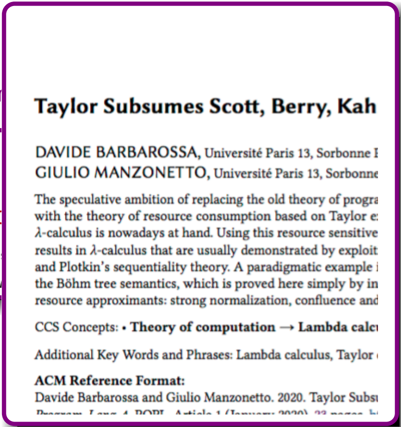

Taylor Subsumes Scott, Berry, Kah

DAVIDE BARBAROSSA, Université Paris 13, Sorbonne
GIULIO MANZONETTO, Université Paris 13, Sorbonne

The speculative ambition of replacing the old theory of progra
with the theory of resource consumption based on Taylor e
λ-calculus is nowadays at hand. Using this resource sensitive
results in λ-calculus that are usually demonstrated by exploit
and Plotkin's sequentiality theory. A paradigmatic example i
the Böhm tree semantics, which is proved here simply by in
resource approximants: strong normalization, confluence and

CCS Concepts: • **Theory of computation → Lambda calc**

Additional Key Words and Phrases: Lambda calculus, Taylor

ACM Reference Format:
Davide Barbarossa and Giulio Manzonetto. 2020. Taylor Subs

The resource $\lambda\mu$-calculus

Define the set $\lambda\mu^{\mathrm{r}}$ of resource $\lambda\mu$-terms:

$$t ::= x \mid \lambda x.t \mid t\,[t,\ldots,t] \mid \mu\alpha._{\beta}|t|$$

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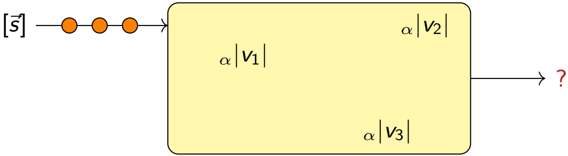
$$(\mu\alpha.{}_{\beta}|t|)[\vec{s}] \to_{\mu} ?$$



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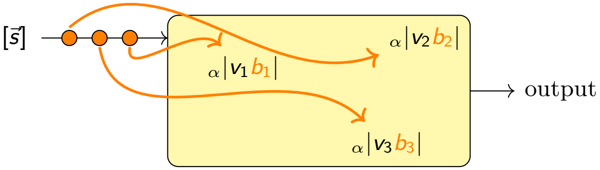

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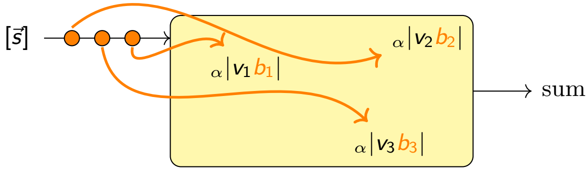
$$(\mu\alpha._{\beta}|t|)[\vec{s}] \to_{\mu} \sum_{b_1 * \cdots * b_k = [\vec{s}]} \mu\alpha._{\beta}|t|\{\dots, {}_{\alpha}|(\cdot) b_i|/_{\alpha|\cdot|^{(i)}}, \dots\}$$



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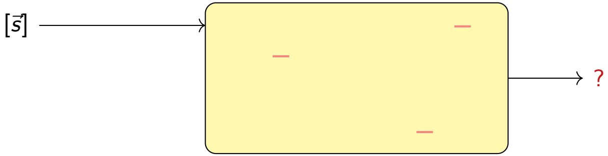

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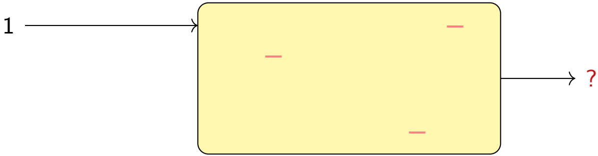

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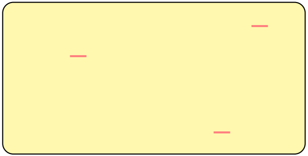

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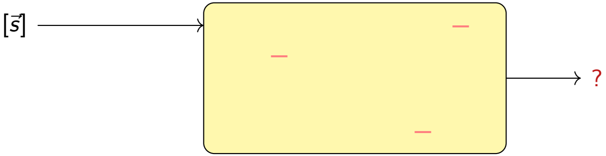

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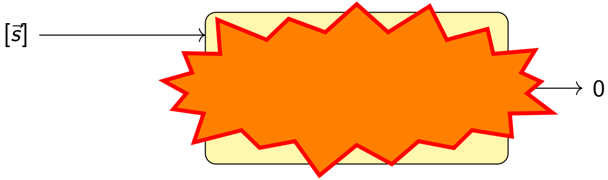

Strong normalization

In λ-reduction: $\sharp(\lambda)$ decreases ✓

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In μ-reduction: $\sharp$ of λ and of μ constant, many new bags, so... ???

$$u := (\mu\alpha._\beta|t|)[\vec{s}] \to_\mu \mu\alpha._\beta|t|\{\ldots, {}_\alpha|(\cdot)b_i|/_{\alpha|\cdot|^{(i)}}, \ldots\}$$

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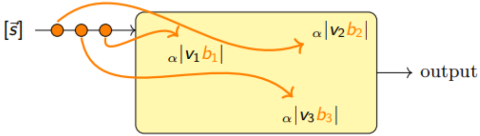

BUT: no more $[\vec{s}]$ + new bags are at a deeper depth inside ${}_{\beta}|t|$

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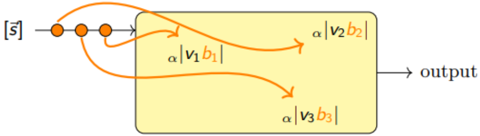

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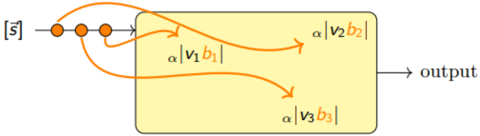

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$\Rightarrow 2\,\sharp(\mu) - \mathrm{depth}(b)$ decreses!

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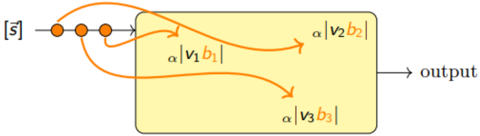

BUT: no more $[\vec{s}]$ + new bags are at a deeper depth inside ${}_\beta|t|$

So: $\mathrm{depth}(b)$ *increases.* Also: $\mathrm{depth}(b) \leq \sharp(\mu)$

$\Rightarrow 2\,\sharp(\mu) - \mathrm{depth}(b)$ decreses!

$$\Rightarrow \mathrm{m}(u) := [\,2\,\sharp_u(\mu) - \mathrm{depth}_u(b) \mid b \text{ bag in } u\,] \in !\mathbb{N} \text{ decreases } ✓$$

Confluence

Technical Lemma

The extentions of the reduction on sums is contextual and remains so also if you include as contexts the linear substitutions.

Local confluence (close all reduction diagrams by Technical Lemma)

The resource $\lambda\mu$-calculus is locally confluent

Corollary (LC + SN + Newman Lemma)

The sums of the resource $\lambda\mu$-calculus are confluent

Qualitative Taylor Expansion

The (support of the full) Taylor expansion is the map $\mathcal{T} : \lambda\mu \to \mathcal{P}(\lambda\mu^{\mathrm{r}})$:

$$
\begin{aligned}
\mathcal{T}(x) &= \{x\} \\
\mathcal{T}(\lambda x.M) &= \{\lambda x.t \text{ s.t. } t \in \mathcal{T}(M)\} \\
\mathcal{T}(MN) &= \{t[s_1, \dots, s_k] \text{ s.t. } k \in \mathbb{N}, t \in \mathcal{T}(M), s_i \in \mathcal{T}(N)\}
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Normalizing Taylor

$$\mathrm{NF}\mathcal{T}(M) := \bigcup_{t \in \mathcal{T}(M)} \mathrm{nf}(t)$$

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Normalizing Taylor

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Taylor normal form order on $\lambda\mu$

Define the partial preorder: $M \leq N$ iff $\mathrm{NF}\mathcal{T}(M) \subseteq \mathrm{NF}\mathcal{T}(N)$

Taylor is well behaving

Monotonicity of contexts

The map $C : \lambda\mu \to \lambda\mu$ (for C context) is monotone w.r.t. $\leq$

Under Taylor, substitutions = linear substitution

$$\mathcal{T}((M)_\alpha N) = \bigcup \langle \mathcal{T}(M) \rangle_\alpha \, !\mathcal{T}(N)$$

Simulation of reduction

If $M \to N$ then:

- for all $s \in \mathcal{T}(M)$ there is $\mathbb{T} \subseteq \mathcal{T}(N)$ s.t. $s \twoheadrightarrow \mathbb{T}$
- for all $s' \in \mathcal{T}(N)$ there is $s \in \mathcal{T}(M)$ s.t. $s \twoheadrightarrow s' + \mathit{something}$

Go to normal form

For all $\mathbb{T} \subseteq \mathcal{T}(M)$ there is N s.t. $M \twoheadrightarrow N$ and $\mathrm{nf}(\mathbb{T}) \subseteq \mathcal{T}(N)$

A non-trivial sensible $\lambda\mu$-theory

Taylor normal form equivalence

Set $M =_{\mathrm{NF}\mathcal{T}} N$ iff $\mathrm{NF}\mathcal{T}(M) = \mathrm{NF}\mathcal{T}(N)$

It is a non trivial $\lambda\mu$-theory

$=_{\mathrm{NF}\mathcal{T}}$ is a congruence (this is Monotonicity);
it is stable on $=_{\lambda\mu\rho}$-classes;
$\mathtt{I} \neq_{\mathrm{NF}\mathcal{T}} \Omega$.

It is sensible

In particular, $\mathrm{NF}\mathcal{T}(M) = \emptyset$ iff M unsolvable.

Where M solvable means that its head-reduction terminates.

Stability

Sufficient conditions for Contexts to be stable under intersections

$$C\llparenthesis \bigcap_{i_1} N_{i_1}, \ldots, \bigcap_{i_k} N_{i_k} \rrparenthesis =_{\mathrm{NF}\mathcal{T}} \bigcap_{i_1,\ldots,i_k} C\llparenthesis N_{i_1}, \ldots, N_{i_k} \rrparenthesis$$

Proved exactly as in λ-calculus. (See Barbarossa, Manzonetto, POPL20).

Stability

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$$C(\!|\bigcap_{i_1} N_{i_1}, \ldots, \bigcap_{i_k} N_{i_k}|\!) =_{\mathrm{NF}\mathcal{T}} \bigcap_{i_1,\ldots,i_k} C(\!| N_{i_1}, \ldots, N_{i_k} |\!)$$

Proved exactly as in λ-calculus. (See Barbarossa, Manzonetto, POPL20).
It crucially relies on the following:

Non-interference Property

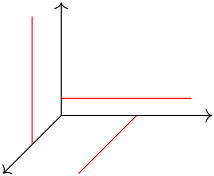
For $t, s \in \mathcal{T}(M)$, if $t \neq s$ then $\mathrm{nf}(t) \cap \mathrm{nf}(s) = \emptyset$

Proof:

Induction on $(\mathrm{m}(t), \mathrm{size}(t)) \in !\mathbb{N} \times \mathbb{N}$.

Perpendicular Lines Property for $\lambda\mu$

If a context $C(\!|\cdot,\ldots,\cdot|\!) : \lambda\mu^n/_{=_{\mathrm{NF}\mathcal{T}}} \to \lambda\mu/_{=_{\mathrm{NF}\mathcal{T}}}$ *is constant on n perpendicular lines, then it must be constant everywhere.*



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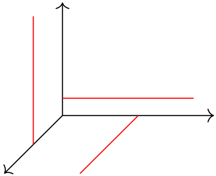

Crucial Lemma

Fix $\vec{z} = z_1,\ldots,z_n$ distinct variables and $t \in \lambda\mu^{\mathrm{r}}$. Then:

$$\left.\begin{array}{l} \mathrm{nf}(t) \neq 0 \\ \\ t \in \mathcal{T}(F) \textit{ for some } F \in \lambda\mu \\ \\ \lambda\vec{z}.F \textit{ constant (mod } =_{\mathrm{NF}\mathcal{T}} \textit{)} \\ \quad \textit{on } n \textit{ perpendicular lines} \end{array}\right\} \quad \Rightarrow \quad z_1,\ldots,z_n \notin t.$$

Induction on $(\mathrm{m}(t), \mathrm{size}(t)) \in\, !\mathbb{N} \times \mathbb{N}$. (See Barbarossa, Manzonetto,

Sequentiality

The $\lambda\mu$-calculus can only implement sequential computations.

Otherwise, we could semidecide "double sovability" in it, which we cannot:

No Parallel-or

There is *no* $\texttt{Por} \in \lambda\mu$ s.t. for all $M, N \in \lambda\mu$

$$\begin{cases} \texttt{Por}\, M\, N & =_{\mathrm{NF}\mathcal{T}} \texttt{True} & \textit{if } M \textit{ or } N \textit{ solvable} \\ \texttt{Por}\, M\, N & \textit{unsolvable} & \textit{otherwise.} \end{cases}$$

By Stability or PLP.

Questions raised:

- Continue developing mathematical study!
- Properties proper to $\lambda\mu$-calculus involving continuations?
- Links with Vaux's differential $\lambda\mu$-calculus?
- Relation to CPS-translations?
- Böhm trees for $\lambda\mu$-calculus or not?
- What about Saurin's $\Lambda\mu$-calculus?
- What does that say about Linear Logic?
- Does that say something about Krivine's Classical Realizability?


That's all Folks!