The resource approximation for the $\lambda\mu$-calculus

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Curry-Howard correspondence

The intuitionistic case:

Intuitionistic proofs express their computational content inside λ-calculus:

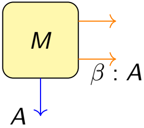
$$M ::= x \mid \lambda x.M \mid MM \qquad (\lambda x.M)N \to M\{N/x\}$$

The classical case: (one possibility among many)

$\lambda\mu$-calculus: classical proofs express their computational content in it

$$M ::= x \mid \lambda x.M \mid MM \mid \mu\alpha.{}_{\beta}|M|$$

The intuition behind $_\beta|\_|$ and $\mu\beta.\_$



M

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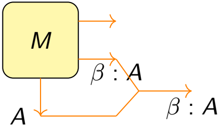
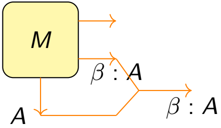


$_\beta|M|$ M returns on auxiliary output β...

The intuition behind $_\beta|\_|$ and $\mu\beta.\_$



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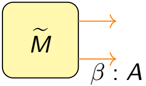
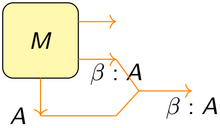


$\widetilde{M}$

The intuition behind $_\beta|\_|$ and $\mu\beta.\_$



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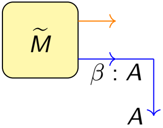


$\mu\beta.\widetilde{M}$ ...declares β the standard output of $\widetilde{M}$

The $\lambda\mu$-calculus (Parigot '92)

Terms

$$M ::= x \mid \lambda x.M \mid MM \mid \mu\alpha.{}_\beta|M|$$

Reduction

$$(\lambda x.M)N \to_\lambda M\{N/x\}$$

$$\mu\alpha.{}_\beta|\mu\gamma.{}_\eta|M|| \to_\rho \mu\alpha.{}_\eta|M|\{\beta/\gamma\}$$

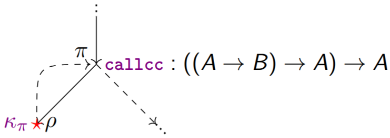
$$(\mu\alpha.{}_\beta|M|)N \to_\mu \mu\alpha.({}_\beta|M|)_\alpha N$$

where $({}_\beta|M|)_\alpha N := {}_\beta|M|\{{}_\alpha|(\cdot)N|/{}_{\alpha}|\cdot|\}$.

It is an impure functional Prog Lang:

Continuations in $\lambda\mu$-calculus

$\texttt{callcc} := \lambda y.\mu\alpha.{}_\alpha|y(\lambda x.\mu\delta.{}_\alpha|x|)|$



Resource $\lambda\mu$-calculus

λ-calculus	$\lambda\mu$-calculus
Differential λ-calculus (Ehrhard-Regnier)	Differential $\lambda\mu$-calculus (Vaux)
Resource λ-calculus	

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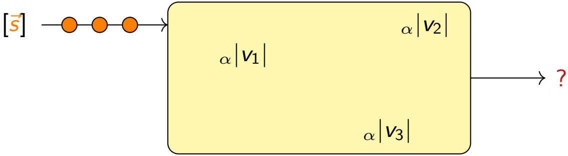
Resource $\lambda\mu$-terms:

$$t ::= \; x \;\mid \lambda x.t \;\mid t\,[t,\ldots,t] \;\mid \mu\alpha.{}_\beta|t|$$

Reduction: $(\lambda x.t)[\vec{s}\,] \to_\lambda t\langle[\vec{s}\,]/x\rangle \qquad \mu\alpha.{}_\beta|\mu\gamma.{}_\eta|t|| \to_\rho \mu\alpha.{}_\eta|t|\{\beta/\gamma\}$

$$(\mu\alpha.{}_\beta|t|)[\vec{s}\,] \to_\mu \text{?}$$



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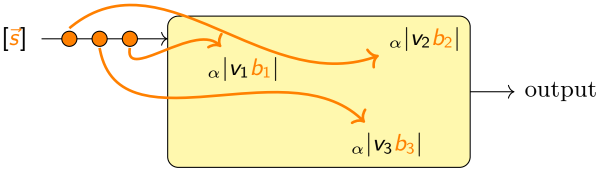
$$(\mu\alpha.{}_{\beta}|t|)[\vec{s}] \to_{\mu} \mu\alpha.{}_{\beta}|t|\{\ldots,\ {}_{\alpha}|(\cdot)b_i|/_{\alpha|\cdot|^{(i)}},\ldots\}$$



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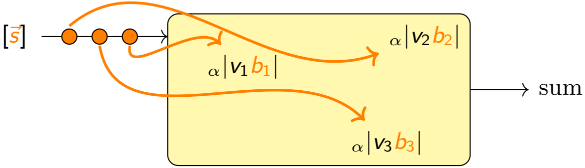
Resource $\lambda\mu$-terms:

$$t ::= x \mid \lambda x.t \mid t\,[t,\dots,t] \mid \mu\alpha.{}_\beta|t|$$

Reduction: $(\lambda x.t)[\vec{s}\,] \to_\lambda t\langle[\vec{s}\,]/x\rangle$ $\qquad \mu\alpha.{}_\beta|\mu\gamma.{}_\eta|t|| \to_\rho \mu\alpha.{}_\eta|t|\{\beta/\gamma\}$

$$(\mu\alpha.{}_\beta|t|)[\vec{s}\,] \to_\mu \sum_{b_1*\cdots*b_k=[\vec{s}\,]} \mu\alpha.{}_\beta|t|\{\dots, {}_\alpha|(\cdot)b_i|/{}_{\alpha|\cdot|^{(i)}}, \dots\}$$



Resource $\lambda\mu$-calculus is well behaved

Linearity

Each resource is used exactly once along a non-annihilating reduction

Strong normalisation

Not immediate

Confluence

Hard:

- Add coefficients: gain contextuality of reduction on sums
- Prove local confluence in the setting with coefficients (treat all critical pairs)
- Show that this entails the confluence of the calculus without coefficients

Qualitative Taylor Expansion

The (support of the full) Taylor expansion is the map $\mathcal{T} : \lambda\mu \to \mathcal{P}(\lambda\mu^{\mathrm{r}})$:

$$\begin{array}{lcl} \mathcal{T}(x) & = & \{x\} \\ \mathcal{T}(\lambda x.M) & = & \{\lambda x.t \text{ s.t. } t \in \mathcal{T}(M)\} \\ \mathcal{T}(MN) & = & \{t[s_1, \ldots, s_k] \text{ s.t. } k \in \mathbb{N}, t \in \mathcal{T}(M), s_i \in \mathcal{T}(N)\} \\ \mathcal{T}(\mu\alpha.{}_\beta|M|) & = & \{\mu\alpha.{}_\beta|t| \text{ s.t. } t \in \mathcal{T}(M)\} \end{array}$$

Taylor transforms substitutions in linear substitution

$\mathcal{T}((M)_\alpha N) = \bigcup \langle \mathcal{T}(M) \rangle_\alpha \mathcal{M}_{\mathrm{fin}}(\mathcal{T}(N))$

Taylor is well behaved

Monotonicity of contexts

The map $C : \lambda\mu \to \lambda\mu$ (for C context) is monotone w.r.t. $\subseteq_{\mathrm{NFT}}$

Simulation property

If $M \to N$ then:

- for all $s \in \mathcal{T}(M)$ there is $\mathbb{T} \subseteq \mathcal{T}(N)$ s.t. $s \twoheadrightarrow \mathbb{T}$
- for all $s' \in \mathcal{T}(N)$ there is $s \in \mathcal{T}(M)$ s.t. $s \twoheadrightarrow s' + \mathit{something}$

Go to normal form

For all $\mathbb{T} \subseteq \mathcal{T}(M)$ there is N s.t. $M \twoheadrightarrow N$ and $\mathrm{nf}(\mathbb{T}) \subseteq \mathcal{T}(N)$

Non-interference property

Let $t, s \in \mathcal{T}(M)$. Then: $\mathrm{nf}(t) \cap \mathrm{nf}(s) \neq \emptyset \Rightarrow t = s$.

A non-trivial sensible $\lambda\mu$-theory

Define M solvable when its head-reduction terminates. Then:
The equivalence equating $\mathrm{NF}(\mathcal{T}(\cdot))$'s is a sensible non-trivial $\lambda\mu$-theory.

Proof:
$\mathrm{NF}\mathcal{T}(M) = \emptyset$ iff M unsolvable ($\Rightarrow$ easy, $\Leftarrow$ not immediate).

Mathematical properties of $\lambda\mu$-calculus

Stability

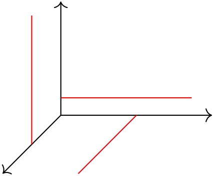
Sufficient conditions for contexts to commute with intersections:

$$C\llparenthesis \bigcap_{i_1} N_{i_1}, \ldots, \bigcap_{i_k} N_{i_k} \rrparenthesis =_{\mathrm{NF}\mathcal{T}} \bigcap_{i_1,\ldots,i_k} C\llparenthesis N_{i_1}, \ldots, N_{i_k} \rrparenthesis$$

Proof: Induction on $\mathcal{M}_{\mathrm{fin}}(\mathbb{N}) \times \mathbb{N}$ (as in Barbarossa-Manzonetto POPL20)

Perpendicular Lines Property

If a context $C\llparenthesis \cdot, \ldots, \cdot \rrparenthesis : \lambda\mu^n/_{=_{\mathrm{NF}\mathcal{T}}} \to \lambda\mu/_{=_{\mathrm{NF}\mathcal{T}}}$ is constant on n perpendicular lines, then it must be constant everywhere.



Proof: Induction on $\mathcal{M}_{\mathrm{fin}}(\mathbb{N}) \times \mathbb{N}$ (as in Barbarossa-Manzonetto POPL20)

Corollary: sequentiality

The $\lambda\mu$-calculus can only implement sequential computations.

Otherwise, it could semi-decide the "double solvability problem", which it cannot:

No Parallel-or

There is *no* $\texttt{Por} \in \lambda\mu$ s.t. for all $M, N \in \lambda\mu$

$$\begin{cases} \texttt{Por}\, M\, N \; =_{\mathrm{NF}\mathcal{T}} \; \texttt{True} & \text{if } M \text{ or } N \text{ solvable} \\ \texttt{Por}\, M\, N \; \text{unsolvable} & \text{otherwise.} \end{cases}$$

Conclusions

Some questions

- Relation to CPS-translations?
- Böhm trees for $\lambda\mu$-calculus?
- Can we do the same for Saurin's $\Lambda\mu$-calculus?

Take home

- We proved results about the mathematics of the $\lambda\mu$-calculus
- Well behaved resource approximation is a powerful technique that you may want to apply to your favourite language !


ANY
QUESTIONS
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