# Tropical Mathematics and the $\lambda$-Calculus

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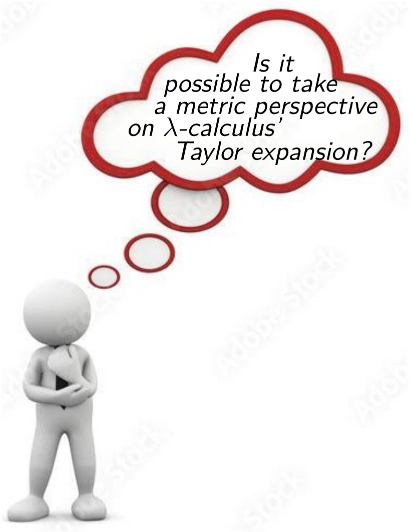
# Motivating question (Ugo da Lago to Pistone & myself)




**Logarithmic gap**

Lipschitz $n\alpha$ vs Polynomial $\alpha^n$

Can they coexist ?

...Yes, in a *tropical* world !

# Linearity

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Two *orthogonal* approaches:

- *Metric approach*: "Duplication as program sensitivity"
  easy terms, difficult typing – types handle duplication through abstraction
- *Differential approach*: "Duplication as linear application"
  difficult terms, easy typing – terms handle duplication through application

# Metric approach: bST$\lambda$C

**Syntax**

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**Feature (sensitive *abstraction*)**

If $\mathtt{M} : !_n A \multimap B$ and $\mathtt{N} : A$, then $\mathtt{MN} : B$ runs with $\mathtt{M}$ calling $\mathtt{N}$ *at most* $n$ times.

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### Intuition

Programs $\mathtt{M} : !_n A \multimap B$ can be seen as *$n$-Lipschitz functions* from a space $A$ to a space $B$.

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**Feature (linear *application*)**

If $\mathtt{M} : A \to B$ and $\mathtt{N} : A$, then $\mathsf{D}^{\mathtt{n}}[\mathtt{M},\mathtt{N}^{\mathtt{n}}]\mathtt{0} : B$ runs with $\mathtt{M}$ calling $\mathtt{N}$ *exactly* $n$ times.

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### Intuition

Programs $\mathsf{D}^{\mathtt{n}}[\mathtt{M},\mathtt{N}^{\mathtt{n}}]\mathtt{0}$ can be seen as *polynomials* and $\mathtt{MN}$ can be *Taylor expanded* as the series $\mathcal{T}(\mathtt{MN}) := \sum_{n\in\mathbb{N}} \frac{1}{n!}\mathsf{D}^{\mathtt{n}}[\mathtt{M},\mathtt{N}^{\mathtt{n}}]\mathtt{0}$.

# Categorical Semantics

Fix a (multi)category $\mathcal{C}$ and give a (multi)functor:

$$\begin{array}{rcll} A & \mapsto & [\![A]\!] & \in Obj(\mathcal{C}) \\ \Gamma \vdash \mathtt{M} : A & \mapsto & [\![\Gamma \vdash \mathtt{M} : A]\!] & \in \mathcal{C}([\![\Gamma]\!], [\![A]\!]). \end{array}$$

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### $Q$-Weighted Relational Semantics

Fix a semiring $Q$. Define the category $Q$Rel as: objects are sets; morphisms from $X$ to $Y$ are matrices with coefficients in $Q$ whose rows are indexed by $Y$ and columns are indexed by $X$, i.e. $Q\mathrm{Rel}(X, Y) := Q^{X \times Y}$.

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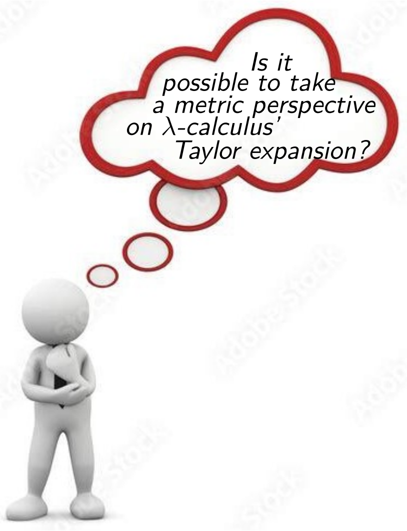


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# Tropical Math

## Tropical semiring = Lawver quantale

- $\mathbb{L} := [0, \infty]$ with addition the inf (neutral element $\infty$) and multiplication the $+$ (neutral element 0).
- In $\mathbb{L}$ we have $n\alpha = \alpha^n$.

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**Theorem (Not surprising!)**

$\mathbb{L}$Rel, *i.e. Q*Rel *for* $Q := \mathbb{L}$, *is a model of the linear-$\lambda$-calculus, of* $\mathrm{bST\lambda C}$, *of* $\mathrm{ST\lambda C}$ *and of* $\mathrm{ST\partial\lambda C}$.

# Metric vs Differential meet at the tropics

Endow $\mathbb{L}^X$ with the usual $\|\_\|_\infty$-norm

**Theorem**

1. $\vdash_{\mathsf{b}\mathrm{ST}\lambda\mathrm{C}} \lambda \mathtt{x}.\mathtt{M} : A \to B$ *gives a tropical polynomial (hence, Lipschitz)* $\mathbb{L}^{[\![A]\!]} \to \mathbb{L}^{[\![B]\!]}$.
2. $\vdash_{\mathrm{ST}\lambda\mathrm{C}} \lambda \mathtt{x}.\mathtt{M} : A \to B$ *gives a locally Lipschitz map* $\mathbb{L}^{[\![A]\!]} \to \mathbb{L}^{[\![B]\!]}$
3. *The Taylor expansion* $\mathcal{T}(\mathtt{M})$ *of* $\mathtt{M}$ *decomposes* $\vdash_{\mathrm{ST}\lambda\mathrm{C}} \lambda \mathtt{x}.\mathtt{M} : A \to B$ *into an* inf *of Lipschitz maps of higher and higher Lipschitz constant.*

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With a probabilistic coin toss instruction $\oplus_p$ of bias $p$, we have results like:

$$[\![\Gamma \vdash \mathtt{M} : \mathrm{Bool}]\!]_1(-\log p, -\log(1-p))$$

gives the *negative log-probability* of any of the *most likely* reduction paths from M to True.

... more to come on that, and much more!

# Grazie !