Errata Corrige – PhD Thesis Manuscript

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In these notes I will correct the errors or unclear parts in my PhD thesis, at least the ones that I spotted until the above date.

Section 3.3

Theorem 3.3.37

In the proof, the following sentence has to be added, just between the last line of page 55 and the first line ("So for all $i = 1, ..., n$ and $j = 1, ..., d_i$, there exist w...") of page 56:

Wlog we can assume $\deg_{\Box_i}(c_{\vec{N}}) \geq 1$ for all i. In fact, if it was $= 0$ for some i, call them $i_1, \ldots, i_k$, then we can consider $c'_{\vec{N}} := c^{\bullet}_{\vec{N}}\langle \Box_1, \ldots, \Box_{i_1-1}, \vec{h}^{i_1}, \Box_{i_1+1}, \ldots, \Box_{i_k-1}, \vec{h}^{i_k}, \Box_{i_k+1}, \ldots, \Box_n \rangle$, for any $h^{i_j} \in \mathcal{T}(M_i)$, $j = 1, \ldots, i_k$.

Then we have

$$c'^{\bullet}_{\vec{N}}\langle \vec{v}^1, \ldots, \vec{v}^{i_1-1}, \vec{v}^{i_1+1}, \ldots, \vec{v}^{i_k-1}, \vec{v}^{i_k+1}, \ldots, \vec{v}^n \rangle$$
$$= c^{\bullet}_{\vec{N}}\langle \vec{v}^1, \ldots, \vec{v}^{i_1-1}, \vec{h}^{i_1}, \vec{v}^{i_1+1}, \ldots, \vec{v}^{i_k-1}, \vec{h}^{i_k}, \vec{v}^{i_k+1}, \ldots, \vec{v}^n \rangle$$
$$= c^{\bullet}_{\vec{N}}\langle \vec{v}^1, \ldots, \vec{v}^n \rangle$$

for some rigid $c'^{\bullet}_{\vec{N}}$ of $c'_{\vec{N}}$.

Now $\deg_{\Box_i}(c'_{\vec{N}}) \geq 1$ for all $i = i_1, \ldots, i_k$, and we can continue the proof with this c' instead of c.

In the case $n = 1$ and $\deg_{\Box}(c_{\vec{N}}) = 0$, this amounts to consider $c'_{\vec{N}} := c^{\bullet}_{\vec{N}}\langle \vec{h} \rangle = c^{\bullet}$, and the rest of the proof is directly bypassed by remarking that $t \in \mathrm{nf}(c'_{\vec{N}}) \subseteq \mathrm{NF}\mathcal{T}(C\langle M \rangle)$ and so we are already done.

Moreover, let me give below examples showing that both the hypotheses "$\mathcal{X} \neq \emptyset$" and "$\mathcal{X}$ upper bounded" are necessary.

For the first, take $\mathcal{X} := \emptyset$ and $C := I$. We have $\mathcal{X}$ upper bounded by any M and $\inf \mathcal{X} = \emptyset =_{\mathcal{T}} \Omega$. The only hypotheses which fails is thus $\mathcal{X} \neq \emptyset$, and we have the following contradiction, if the theorem was true: $I =_{\mathcal{T}} C\langle \Omega \rangle =_{\mathcal{T}} \inf_{N \in \emptyset} C\langle N \rangle = \inf \emptyset = \emptyset =_{\mathcal{T}} \Omega$.

For the second, take $\mathcal{X} := \{0, 1\}$ (Church numerals) and $C := \Box\, I\, I$. We have $\mathcal{X} \neq \emptyset$ and $\inf \mathcal{X} = \emptyset =_{\mathcal{T}} \Omega$. The only hypotheses which fails is thus $\mathcal{X}$ being upper bounded (since 0 and 1 are distinct head normal forms, the root of their Bohm trees are different and thus their Bohm trees do not share any common upper bound, and thus the same must hold for $\leq_{\mathcal{T}}$, since the two partial preorders are the same). We have the following contradiction, if the theorem was true: $\Omega =_{\mathcal{T}} C\langle \Omega \rangle =_{\mathcal{T}} \inf\{C\langle 0 \rangle, C\langle 1 \rangle\} =_{\mathcal{T}} \inf\{I, I\} =_{\mathcal{T}} I$.

Section 3.5

Lemma 3.5.5

In the statement, the sentence:

"In particular, one has therefore $\mathcal{T}(M) \subseteq \mathcal{T}(\mathrm{BT}(M))$."

should be instead:

"In particular, one has therefore $\mathcal{T}(M) \cap \mathrm{Normal} \subseteq \mathcal{T}(\mathrm{BT}(M))$."

Note also that this is not the same as $\mathrm{NF}\mathcal{T}(M)$.

Lemma 3.5.11(2)($\Rightarrow$)

The proof is wrong, but the result is correct. The correct proof is the following:

Proof. Let $\overline{P} \in \mathrm{NF}\mathcal{T}(M)$. We have to show that $P \in \mathcal{A}(M)$. By definition, $\overline{P} \in \mathrm{nf}(t)$, for some $t \in \mathcal{T}(M)$. By Lemma 3.3.25, there is N s.t. $M \twoheadrightarrow N$ and $\overline{P} \in \mathcal{T}(N)$. Lemma 3.5.5, $\overline{P} \in \mathcal{T}(Q)$, for some $Q \in \mathcal{A}(N)$. By Lemma 3.5.11(1), $P \sqsubseteq Q$. Since $\mathcal{A}(\cdot)$ is closed w.r.t. $\sqsubseteq$, we have $P \in \mathcal{A}(N)$. But since (by definition) $\mathcal{A}(\cdot)$ is closed by antireduction, we get $P \in \mathcal{A}(M)$. □

Lemma 3.5.31 and Lemma 3.5.32

The proof and the statements are correct, but the hypotheses "Let M be solvable" can be dropped from the statement (and thus from the inductive hypothesis in the proof), since if M is unsolvable then there are no t in $\mathrm{NF}\mathcal{T}(M)$, and therefore the claim still trivially holds.

Lemma 3.5.32

In the last line of the proof, it should be:

"$u^i_j \in \mathcal{T}(P_{u^i_j}) \subseteq \mathcal{T}(\bigsqcup_j P_{u^i_j})$, where $\bigsqcup_j P_{u^i_j}$ exists for all i because the finitely many $P_{u^i_1}, \ldots, P_{u^i_{k_i}}$ all belong to $\mathcal{A}(M_i)$ by Lemma 3.5.31. So we conclude $t \in \mathcal{T}(\lambda\vec{x}.y(\bigsqcup_j P_{u^1_j}) \cdots (\bigsqcup_j P_{u^k_j}))$".

Lemma 3.5.33

Thea arguments in the proof are correct but it could be more clearly written as follows, were I also explicit some steps which were left implicit.

In particular, we need the following two Lemmas[1]:

Lemma 0.1 (Lemma Bonus1)**.** *Let $Q \in \mathcal{A}pp$. Then $\mathcal{T}(Q) \subseteq \mathcal{T}(Q\{\Omega/\bot\})$.*

Proof. Straightforward induction on $Q \in \mathcal{A}pp$. □

Lemma 0.2 (Lemma Bonus2)**.** *Let $Q \in \mathcal{A}pp$. Then $\mathcal{A}(C\langle Q\{\Omega/\bot\}\rangle) \subseteq \mathcal{A}(C\langle Q\rangle)$.*

Proof. Let $P \in \mathcal{A}(C\langle Q\{\Omega/\bot\}\rangle)$. Then there is a reduction $C\langle Q\{\Omega/\bot\}\rangle \twoheadrightarrow N \sqsupseteq P$. But then we can color the substituted Ω's and carry on the same exact reduction steps as above starting from $C\langle Q\{\hat{\Omega}/\bot\}\rangle\{\bot/\hat{\Omega}\} = C\langle Q\rangle$, where the $\hat{\Omega}$'s are the coloured Ω's. Indeed, in the reduction, an Ω can never been used to create or erase redexes; it can only contribute by reducing to itself, in which case we can skip this reduction step. Therefore, the coloured Ω's, which already appear from the beginning of the reduction, could have only been moved around the terms in

[1] They use the notion of Bohm approximants $\mathcal{A}pp$ of Definition 2.2.3.

the reduction, or being erased, or reducing to itself. Therefore, if we put $\bot$ instead of them, we can do exactly the same reduction steps, by skipping the ones where we reduced a coloured Ω to itself. Therefore we obtain a reduction $C\langle Q\rangle \twoheadrightarrow N\{\bot/\hat{\Omega}\}$. But $N\{\bot/\hat{\Omega}\} \sqsupseteq P$, because $P \sqsubseteq N$ means that some $\bot$'s in P have been replaced by subterms of N, we can plug the subterms $N\{\bot/\hat{\Omega}\}$ of $N\{\bot/\hat{\Omega}\}$ for the same $\bot$ in P and obtain $N\{\bot/\hat{\Omega}\}$. Therefore, $P \in \mathcal{A}(C\langle Q\rangle)$. $\square$

Now we can prove Lemma 3.5.33.

Proof of Lemma 3.5.33. We have to show the existence of a $Q \in \mathcal{A}(M)$ s.t. $\mathcal{A}(P) \subseteq \mathcal{A}(C\langle Q\rangle)$.

Since P is normal, $\mathcal{A}(P) = \{P' \mid P' \sqsubseteq P\}$ and this is a finite set $\{P_1, \ldots, P_k\} \cup \{\bot\}$, where all the $P_i \neq \bot$.

Since $\bot \in \mathcal{A}(C\langle Q\rangle)$, we only have to show that $P_i \in \mathcal{A}(C\langle Q\rangle)$ for $i = 1, \ldots, k$.

For each i, since $\bot \neq P_i \in \mathcal{A}(P) \subseteq \mathcal{A}(C\langle M\rangle)$ by hypothesis, by Lemma 3.5.11(2)($\Leftarrow$) we have $\overline{P_i} \in \mathrm{NF}\mathcal{T}(C\langle M\rangle)$. Therefore, $\overline{P_i} \in \mathrm{nf}(c_i^\bullet\langle \vec{s}^i\rangle)$, where $c_i \in \mathcal{T}(C)$ and $\vec{s}^i = \langle s_1^i, \ldots, s^i_{\deg_\square c_i}\rangle$ with all the elements of the list belonging to $\mathcal{T}(M)$. By confluence, we have thus

$$\overline{P_i} \in \mathrm{nf}(c_i^\bullet\langle \mathrm{nf}(\vec{s}^i)\rangle). \tag{1}$$

Let us consider now the *finite* set

$$\mathcal{Q} := \bigcup_{i=1}^{k} \bigcup_{j=1}^{\deg_\square c_i} \mathrm{nf}(s_j^i) \subseteq \mathrm{NF}\mathcal{T}(M).$$

By Lemma 3.5.31, we have $P_u \in \mathcal{A}(M)$ for all $u \in \mathcal{Q}$. But since $\mathcal{Q}$ is finite and $\mathcal{A}(M)$ is directed, then in $\mathcal{A}(M)$ there exists the sup

$$Q := \bigsqcup_{u \in \mathcal{Q}} P_u \in \mathcal{A}(M)$$

of finitely many elements of $\mathcal{A}(M)$.

Now, fix i.

We remark that if both $\square \in c_i$ and $\mathrm{nf}(s_j^i) = \emptyset$ for some $j \in \{1, \ldots, \deg_\square c_i\}$, then $c_i^\bullet\langle \mathrm{nf}(\vec{s}^i)\rangle = \emptyset$ and therefore by (1) we would have a contradiction. Therefore, for each i, we have only two cases: either $\deg_\square c_i = 0$; or $\deg_\square c_i \geq 1$ and $\mathrm{nf}(s_j^i) \neq \emptyset$ for all $j \in \{1, \ldots, \deg_\square c_i\}$.

In the first case we have $c_i^\bullet\langle \vec{s}^i\rangle = c_i^\bullet \in \mathcal{T}(C\langle Q'\rangle)$ for all term Q'. In particular, we can choose $Q' := Q\{\Omega/\bot\}$. So $\overline{P_i} \in \mathrm{NF}\mathcal{T}(C\langle Q\{\Omega/\bot\}\rangle)$.

In the second case, by (1), for all $j \in \{1, \ldots, \deg_\square c_i\}$ there must be $u_j^i \in \mathrm{nf}(s_j^i)$ s.t. $\overline{P_i} \in \mathrm{nf}(c_i^\bullet\langle \vec{u}^i\rangle)$. But then we have: $u_j^i \in \mathcal{T}(P_{u_j^i}) \subseteq \mathcal{T}(Q) \subseteq \mathcal{T}(Q\langle\Omega/\bot\rangle)$ for all j; the first inclusion is by Lemma 3.5.32; the second inclusion is by Lemma 3.5.3 and because $u_j^i \in \mathcal{Q}$ and so $P_{u_j^i} \sqsubseteq Q$ by definition of Q; the last inclusion is by Lemma Bonus1. Therefore $\overline{P_i} \in \mathrm{NF}\mathcal{T}(C\langle Q\{\Omega/\bot\}\rangle)$.

So in both cases we have $\overline{P_i} \in \mathrm{NF}\mathcal{T}(C\langle Q\{\Omega/\bot\}\rangle)$. But by Lemma 3.5.11(2)($\Rightarrow$), we have $P_i \in \mathcal{A}(C\langle Q\{\Omega/\bot\}\rangle)$, and by Lemma Bonus2 we obtain $P_i \in \mathcal{A}(C\langle Q\rangle)$, which is the thesis.

$\square$