

Resource approximation for the $\lambda\mu$ -calculus

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Abstract

The $\lambda\mu$ -calculus plays a central role in the theory of programming languages as it extends the Curry-Howard correspondence to classical logic. A major drawback is that it does not satisfy Böhm's Theorem and it lacks the corresponding notion of approximation. On the contrary, we show that Ehrhard and Regnier's Taylor expansion can be easily adapted, thus providing a resource conscious approximation theory. This produces a sensible $\lambda\mu$ -theory with which we prove some advanced properties of the $\lambda\mu$ -calculus, such as Stability and Perpendicular Lines Property, from which the impossibility of parallel computations follows.

CCS Concepts: • **Theory of computation** → **Lambda calculus**; *Linear logic*; Rewrite systems.

Keywords: $\lambda\mu$ -calculus, Taylor expansion, resource approximation, linear logic, λ -calculus

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1 Introduction

1.1 Motivation

1.1.1 Curry-Howard correspondence for classical logic.

The celebrated *Curry-Howard correspondence* states that a class of programs, written in a suitable programming language, and intuitionistic logic proofs, written in an suitable formal system, are the same mathematical objects. The typical suitable programming language is λ -calculus, and the typical suitable formal system is intuitionistic natural deduction NJ; under this correspondence, the simply typed

λ -calculus is identified with NJ. A natural question is what happens for *classical* logic proofs and whether it is possible to find such a correspondence at all. In the 90's, several ways for generalising such a correspondence to this framework appeared, starting from [13] where Griffin suggests to type control operators (such as Scheme's `callcc` or Felleisen's `C` operator) with Peirce's law. One of the most notable ones is the $\lambda\mu$ -calculus, introduced by Parigot in [25], which has the advantage of allowing the correspondence to take the exact same form as in the intuitionistic case: just like λ -calculus is the Turing-complete programming language in which intuitionistic logic expresses its computational content, $\lambda\mu$ -calculus is the one expressing the computational content of classical logic. From the programming viewpoint, the big difference between λ -calculus and $\lambda\mu$ -calculus is that the former is a *purely functional* language, while the latter is *impure*, due to the possibility of encoding control operators in it — like the already mentioned `callcc` or `C`. From the point of view of, e.g., Classical realizability [19], this corresponds to the backtracking mechanism related to classical reasoning.

1.1.2 Taylor expanding programs.

Just before the 90's another major discovery in logic and computer science appeared: Girard's linear logic [12]. This opened a whole new field of research, in which the common line is the deep role reserved to *resources* in a computation/proof. Linear logic allowed Ehrhard and Regnier to discover an astonishing correspondence between linearity in analysis and linearity in computer science, that is formalized in the *differential λ -calculus* [9] (and differential interaction nets [10]). It is possible to Taylor expand programs/proofs by — as in analysis — an infinite series of approximants weighted via the usual factorial coefficients. This is usually called the “full” or “quantitative” Taylor expansion. However, it turns out that even if we do not consider the coefficients, we still obtain a meaningful theory of program approximation: it is usually called the “qualitative” Taylor expansion, and will play a central role in the present article. Under the assumption of having an idempotent sum, the Taylor expansion is no longer a series but becomes a set, and the approximants can be written in a simple “target language”, called *resource calculus* (very similar to Boudol's calculus with multiplicities [4]). In [1], it is shown that all the fundamental results in the so called *approximation theory* of λ -calculus

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(Monotonicity, Genericity, Continuity, Stability, Perpendicular Lines Property), usually achieved via labelled reductions and Böhm trees [2, Chapter 14], can be actually proven – in an arguably more satisfactory way – via (qualitative) Taylor expansion; in other words, in λ -calculus, resource approximation (which is at the basis of Taylor expansion) “subsumes” Böhm trees approximation, and answers positively to a proposal expressed in [9] where Ehrhard and Regnier mention that: “*Understanding the relation between the term and its full Taylor expansion might be the starting point of a renewing of the theory of approximation.*”

1.1.3 The content of this paper. The aforementioned “approximation results” are in fact at the basis of a *mathematical study* of λ -calculus, deep from the conceptual, mathematical and computational viewpoint. A natural question is then: what about for other programming languages? There are many works which extend the notion of (qualitative as well as quantitative) Taylor expansion to other programming languages ([5, 6, 17, 21, 31]), usually concentrating on its relations with normalisation. Not always easy is, then, *applying* it to actually study the properties of the source language. This is maybe due to the fact that, unlike in the long time studied λ -calculus, for other languages one does not really know what a “mathematical theory of it” should look like. In the present work, we tackle the case of $\lambda\mu$ -calculus for which, to the best of our knowledge, the problem of directly defining a Taylor expansion has never been priorly considered. In this sense, our work can be seen as a continuation of the above mentioned series of papers, and may be related to [18] where the authors study non-idempotent intersection and union types for $\lambda\mu$ -calculus.

We propose here to reverse the above proposal of Ehrhard and Regnier and *start* by defining a resource sensitive version and a Taylor expansion for $\lambda\mu$ -calculus, trying then to *use* this approximation machinery to prove – following [1] – mathematical properties of the language.

A notable work has to be mentioned: in [30], Vaux-Auclair defines a full differential $\lambda\mu$ -calculus following the steps of [9]. Its version takes coefficients into account, and as always this raises a series of non-trivial problems to handle. However, he does not define a Taylor expansion nor does he apply those tools to find properties of the language. The present work can thus be seen also as a continuation of Vaux-Auclair’s one.

There are, from our point of view, several reasons for considering the $\lambda\mu$ -calculus: first of all, from the Curry-Howard point of view, it is the natural “successor” of λ -calculus. Moreover, it is a standard reference for the study of control operators in functional languages. Yet, there are just few attempts to really study its mathematical theory, and the state of the art is not comparable with the well-established one for λ -calculus. For example, Laurent in [23] makes the following observation: “*Models of the simply typed λ -calculus, of the*

untyped λ -calculus and of the simply typed $\lambda\mu$ -calculus are well understood, but what about models of the untyped $\lambda\mu$ -calculus? As far as we know, this question has been almost ignored.” With the same motivation, we look at the other major part which constitutes a mathematical theory of a programming language, namely the *theory of approximation*. In this sense, the present work can be seen as a continuation of [23].

Other points in relation with Krivine’s classical realizability, proof-nets, CPS-translations and Saurin’s $\Lambda\mu$ -calculus will be mentioned in the conclusions.

The article is organised as follows: in Section 2 we define the resource $\lambda\mu$ -calculus and prove that it is strongly normalising and confluent (Corollaries 2.13 and 2.30). In Section 3 we define the qualitative Taylor expansion and prove its main properties, which give rise to a non-trivial sensible “ $\lambda\mu$ -theory” (Corollary 3.15). In Section 4 we apply these approximation tools to prove two important results: Stability (Theorem 4.1) and Perpendicular Lines Property (Theorem 4.4). As a consequence, we obtain the non-representability of *parallel-or* in $\lambda\mu$ -calculus (Corollaries 4.2 and 4.5).

1.2 Quick overview of $\lambda\mu$ -calculus

We briefly recall the definition of the $\lambda\mu$ -calculus, and introduce some basic notions and notation.

Definition 1.1. *Fix a countable set whose elements are called variables and a disjoint countable set whose elements are called names. The set $\lambda\mu$ of $\lambda\mu$ -terms is generated by the following grammar:*

$$M ::= x \mid \lambda x.M \mid MM \mid \mu\alpha.\beta|M|$$

(for x a variable and α, β names) in which, as usual, λ binds x in M as well as μ binds α in $\beta|M|$. Terms are considered up to renaming of bound variables and names.

Despite not being actual subterms, words of shape $\alpha|M|$ are called *named terms*¹. M is said to be *named under α* .

The k -contexts (also called *multihole-contexts* when k is generic) $C = C\{\xi_1, \dots, \xi_k\}$ are defined as expected by:

$$C ::= x \mid \xi_i \mid \lambda x.C \mid CC \mid \mu\alpha.\beta|C|$$

where $\{\xi_1, \dots, \xi_k\}$ is a new set whose elements are called *holes*. 1-contexts are simply called *contexts*. A context with exactly one occurrence of the hole is called *single-hole*, and as usual satisfy: $C ::= \xi_i \mid \lambda x.C \mid CM \mid MC \mid \mu\alpha.\beta|C|$.

¹Historically named terms are written as $[\alpha]M$, as in [25]. But this notation has to be given up since the use of square brackets is already imperatively taken by the *finite multisets*, which we will encounter constantly in the following. Another notation, used in [28], is to write $M\alpha$. However in our framework we find this notation not clear. The notation $\alpha|M|$ should, instead, clearly show what is inside a “naming” and what is not.

Definition 1.2. The reduction relation \rightarrow of $\lambda\mu$ -calculus is the contextual closure² of the union $\rightarrow_{\text{base}}$ of:

$$\begin{aligned} (\lambda x.M)N &\rightarrow_{\lambda} M\{N/x\} \\ (\mu\alpha.\beta|M|)N &\rightarrow_{\mu} \mu\alpha.(\beta|M|)_{\alpha}N \\ \mu\gamma.\alpha|\mu\beta.\eta|M|| &\rightarrow_{\rho} \mu\gamma.(\eta|M|\{\alpha/\beta\}) \end{aligned}$$

where $M\{N/x\}$ is the usual capture-free substitution of N for all free occurrences of x in M , $_{\eta}|M|\{\alpha/\beta\}$ replaces α for all the free occurrences of β in $_{\eta}|M|$, and $(M)_{\alpha}N$ is given by:

$$\begin{aligned} (x)_{\alpha}N &:= x \\ (\lambda x.M)_{\alpha}N &:= \lambda x.(M)_{\alpha}N \\ (MP)_{\alpha}N &:= ((M)_{\alpha}N)((P)_{\alpha}N) \\ (\mu\beta.\gamma|M|)_{\alpha}N &:= \mu\beta.\gamma|(M)_{\alpha}N| \text{ (if } \gamma \neq \alpha) \\ (\mu\beta.\alpha|M|)_{\alpha}N &:= \mu\beta.\alpha|((M)_{\alpha}N)N|. \end{aligned}$$

We denote by $\equiv_{\lambda\mu\rho}$ the equivalence induced by \rightarrow on $\lambda\mu$.

The operation $(M)_{\alpha}N$ coincides with the substitution $M\{\alpha|(\cdot)N|/\alpha|\cdot|\}$: every named subterm $_{\alpha}|\cdot|$ of M gets substituted with the named term $_{\alpha}|\cdot|N$. Nevertheless, “morally”, it is an application: each subterm of M named under α receives a copy of N to be applied to. This is why we chose the notation “ $(M)_{\alpha}N$ ”, which is reminiscent of the application of M to N , and the term $(M)_{\alpha}N$ is called the *named application of M to N through α* . Such notation is due to [30].

The reduction λ is the usual β -reduction (which we call “ λ ” in order to avoid confusion with names). The reduction ρ is just a renaming of names. The novelty is the μ -reduction which, in the following section, we are going to “linearise”. There are many reductions that one can consider on the $\lambda\mu$ -calculus; we chose to stick to those three because they are the ones considered in the original paper [25].

Theorem 1.3. The $\lambda\mu$ -calculus $(\lambda\mu, \rightarrow)$ is confluent.

Proof. See proof of Theorem 4.1 of [26]. \square

Lemma 1.4. Every $\lambda\mu$ -term M has the following shape:

$$M = \lambda\vec{x}_1.\mu\alpha_1.\beta_1|\dots|\lambda\vec{x}_k.\mu\alpha_k.\beta_k|R\vec{Q}||$$

where R is either a variable, or a λ -redex or a μ -redex; furthermore, R , \vec{Q} , k , \vec{x}_i and α_i are unique. R is called the head redex of M if it is a $\lambda\mu$ -redex, and it is called the head variable of M otherwise. The sequence $\lambda\vec{x}_1.\mu\alpha_1.\beta_1|\dots|\lambda\vec{x}_k.\mu\alpha_k.\beta_k|*||$ is called the head of M . Therefore, every $\lambda\mu\rho$ -normal $\lambda\mu$ -term M has a head variable, has no ρ -redexes in its head and (with the previous notations) \vec{Q} are $\lambda\mu\rho$ -normal $\lambda\mu$ -terms.

Other than what we already said in the introduction, we will not add more explanations of the logical and programming meaning of this calculus. Let us just add here the encoding of **callcc** in $\lambda\mu$ -calculus: $\text{callcc} := \lambda y.\mu\alpha.\alpha|y(\lambda x.\mu\delta.\alpha|x)|$.

²The contextual closure of a binary relation \mathcal{R} is the binary relation given by the set: $\{(C|M|), C|(N)|) \mid MRN \text{ and } C(\cdot)$ single hole context $\}$.

2 Resource $\lambda\mu$ -calculus

Recall that a *multiset* A on a set X is a map from X to \mathbb{N} . We use a multiplicative notation: the empty multiset is denoted with 1 and the union of two multisets A, B is denoted with $A * B$. The set of multisets on a set X is a monoid w.r.t. $*$, with neutral element 1. We denote with $!X$ the set of *finite multisets* on X , that is, multisets A with $X - A^{-1}(0)$ finite. Such an A will be as usual written as $A = [a_1, \dots, a_k]$, with $A(a_i)$ repetitions for each a_i . We will sometimes write $m \pm A$ for $[m \pm a_1, \dots, m \pm a_k]$ if $m \pm a_i$ happens to be defined.

Definition 2.1. The set $\lambda\mu^f$ of resource $\lambda\mu$ -terms is given by:

$$t ::= x \mid \lambda x.t \mid t_0[t_1, \dots, t_n] \mid \mu\alpha.\beta|t|$$

where $[t_1, \dots, t_n] \in !\lambda\mu^f$ ($n \geq 0$), and it is called a bag. Resource terms are considered up to renaming of bound variables and names. Resource-contexts are defined as expected. For v a variable or a name, the degree $\text{deg}_v(t) \in \mathbb{N}$ of v in t , is defined as the number of free occurrences of v in t .

The meaning of a resource sensitive application $(\lambda x.t)[\vec{u}]$ is to non-deterministically choose a way to associate each resource in the bag with exactly one occurrence of the argument x in t . It is thus natural to consider (formal) sums. If this association cannot be done without erasing or duplicating resources, then it annihilates to the empty sum 0. The operational semantics of a resource sensitive application $(\mu\alpha.\beta|t|)[\vec{u}]$ will be discussed in Definition 2.5.

Definition 2.2. Call $2\langle\lambda\mu^f\rangle$ the free module generated by $\lambda\mu^f$ over the boolean semiring, which simply means the set of the formal sums of finitely many $\lambda\mu^f$ -terms, quotiented by commutativity, idempotency and associativity of $+$. An element of $2\langle\lambda\mu^f\rangle$ will be called a sum (in fact, it is just a finite subset of $\lambda\mu^f$). We extend the constructors of $\lambda\mu^f$ to $2\langle\lambda\mu^f\rangle$ by linearity, setting:

$$\left(\sum_{i_0} t_{i_0}\right) \left[\sum_{i_1} t_{i_1}, \dots, \sum_{i_n} t_{i_n}\right] := \sum_{i_0, \dots, i_n} t_{i_0}[t_{i_1}, \dots, t_{i_n}]$$

and analogous for $\lambda x.\sum_i t_i$ and $\mu\alpha.\beta|\sum_i t_i|$. We denote with 0 the empty sum. It is the neutral element for $+$ and the annihilating element for the above constructors (i.e. when it appears as any subterm, the whole term becomes 0).

Let us define now a reduction in $\lambda\mu^f$ (or, better said, in $2\langle\lambda\mu^f\rangle$). For this, we will need to divide a multiset into a certain number of “blocks”. This notion already exists in the literature of combinatorics (see for example [3]).

Definition 2.3. A partition (resp. weak partition) of a multiset $[\vec{u}]$ is a multiset $[[\vec{v}_1], \binom{k \geq 1}{}, [\vec{v}_k]]$ of non empty (resp. possibly empty) multisets such that $[\vec{u}] = [\vec{v}_1] * \dots * [\vec{v}_k]$. A composition (resp. weak composition - w.c. for short) of a multiset $[\vec{u}]$ is a tuple $([\vec{v}_1], \dots, [\vec{v}_k])$ of multisets s.t. $[[\vec{v}_1], \dots, [\vec{v}_k]]$ is a partition (resp. weak partition) of $[\vec{u}]$.

Observe that the empty bag 1 admits no partitions but admits infinite weak partitions: they are the multisets of shape $[1, \dots, 1]$ ($h \geq 1$ times 1). Here are some other examples: the set of all the weak partitions of the bag $[x]$ is $\{[[x]], [[x], 1], [[x], 1, 1], \dots\}$. The set of all weak partitions of $[x, x]$ is $\{[[x, x]], [[x], [x]], [[x, x], 1], [[x], [x], 1], [[x, x], 1, 1], [[x], [x], 1, 1], \dots\}$.

Definition 2.4. Let $t \in \lambda\mu^r$ and $[\vec{u}] \in !\lambda\mu^r$. The linear substitution $t\langle u_1, \dots, u_k/x \rangle \in 2\langle \lambda\mu^r \rangle$ is defined, as usual, in Figure 1. In order to linearise the μ -reduction we introduce the linear named application $\langle t \rangle_\alpha[\vec{u}] \in 2\langle \lambda\mu^r \rangle$, defined in Figure 2³.

Remark that, thus, if $\deg_\alpha(t) = 0$ then $\langle t \rangle_\alpha 1 := t$ and $\langle t \rangle_\alpha[v, \vec{u}] := 0$; if $\deg_\alpha(t) =: d \neq 0$ then: $\langle t \rangle_\alpha[\vec{u}]$ is the sum $\sum t \{ \alpha | (\cdot) [\vec{s}^1] /_{\alpha | \cdot |^{(1)}}, \dots, \alpha | (\cdot) [\vec{s}^d] /_{\alpha | \cdot |^{(d)}} \}$, where the sum is taken over all $([\vec{s}^1], \dots, [\vec{s}^d])$ w.c. of $[\vec{u}]$ of length d and $\alpha | \cdot |^{(1)}, \dots, \alpha | \cdot |^{(d)}$ is any fixed enumeration of the occurrences of α in t .

Definition 2.5. Define a reduction $\rightarrow_r \subseteq \lambda\mu^r \times 2\langle \lambda\mu^r \rangle$ as the resource-context closure of the union $\rightarrow_{\text{base}^r}$ of:

$$(\lambda x.t)[\vec{u}] \rightarrow_{\lambda^r} t\langle [\vec{u}]/x \rangle \quad \mu\gamma.\alpha | \mu\beta.\eta | t | \rightarrow_{\rho^r} \mu\gamma.\langle \eta | t | \{ \alpha/\beta \} \\ (\mu\alpha.\beta | t |)[\vec{u}] \rightarrow_{\mu^r} \mu\alpha.\langle \beta | t | \rangle_\alpha[\vec{u}].$$

We extend it to all $2\langle \lambda\mu^r \rangle \times 2\langle \lambda\mu^r \rangle$ setting:

$$\rightarrow_r := \{ (t + \mathbb{S}, \mathbb{T} + \mathbb{S}) \mid t \rightarrow_r \mathbb{T} \text{ and } t \notin \mathbb{S} \}.$$

Observe that the analogue of Lemma 1.4 holds for $\lambda\mu^r$ -terms (in particular we will use the notion of head variable/redex).

The work [1] is an example of how a resource calculus can be useful, as it enjoys strong properties such as linearity, strong normalisation and confluence. In the resource λ -calculus the last two properties are easy; as we are going to see, in our case they are more involved.

2.1 Strong normalisation

With \rightarrow_{λ^r} we erase exactly one λ , with \rightarrow_{ρ^r} we erase exactly one μ . With \rightarrow_{μ^r} however, the situation is more subtle: we are not creating nor erasing λ 's or μ 's (which remain thus in constant number), but we are eventually making the reduct grow by creating an arbitrarily large number of new applications. However, in order to pass from the μ -redex $(\mu\alpha.\beta | t |)[\vec{u}]$ to a reduct $t' \in \mu\alpha.\langle \beta | t | \rangle_\alpha[\vec{u}]$, we: first, decompose $[\vec{u}]$ in several blocks; then, erase $[\vec{u}]$; finally, put each block *inside* a certain named subterm of $\beta | t |$. We replaced thus a bag with many new bags which are at a “deeper depth”. As we will see in Remark 2.7, it will be immediate to recognize that actually this depth is necessary bounded by the number of μ -occurrences in the term, which is invariant under \rightarrow_{μ^r} , so the former subtracted to the latter should decrease. Remark

³The induction takes into account also the case of named terms $\eta | t |$; this is done for technical reasons.

that in the case $[\vec{u}] = 1$, $\deg_\mu(\beta | t |) = 0$ we do not create new applications but we simply erase one already existing one, so we have to make sure our measure decreases in this case as well.

Definition 2.6. Let t be a $\lambda\mu$ -term and let b be an occurrence of a bag or of a subterm of t . The depth $d_t(b) \in \mathbb{N}$ of b in t is the number of named subterms of t containing b .

Remark 2.7. By definition of the grammar of the $\lambda\mu$ -calculus there are as many named subterms of t as μ -abstractions in t , i.e. $\deg_\mu(t)$. So we must have: $d_t(b) \leq \deg_\mu(t)$.

Definition 2.8. Define the multiset measure $m(t) \in !\mathbb{N}$ of a $\lambda\mu^r$ -term t as:

$$m(t) := \deg_\mu(t) - [d_t(b) \mid b \text{ occurrence of bag in } t].$$

Remark 2.7 assures that $m(t) \in !\mathbb{N}$ (and not in $!\mathbb{Z}$). This is crucial because it allows us to reason by induction w.r.t. the multiset order on it. The measure $m(\cdot)$ is “almost” the good one for strong normalization:

Proposition 2.9. If $t \rightarrow_{\mu^r} t' + \mathbb{T}$ then $m(t) > m(t')$.

Proof. If $t \rightarrow_{\mu^r} t' + \mathbb{T}$ then $t = c((\mu\alpha.\beta | s |)b_0)$ and $t' = c(h)$ with $h \in \mu\alpha.\langle \beta | s | \rangle_\alpha b_0$ and c a single-hole resource context. Call $k := \deg_\mu(t) = \deg_\mu(t')$ and consider $\deg_\alpha(\beta | s |) \in \mathbb{N}$. There are two cases:

- Case $\deg_\alpha(\beta | s |) = 0$. By definition of \rightarrow_{μ^r} this is possible only if $b_0 = 1$ (otherwise $t \rightarrow_{\mu^r} 0$) and $h = \mu\alpha.\beta | s |$. So in t there are the exact same occurrences of bags as in t' and they are at the same depth, except for b_0 which is in t but not in t' . This means that $m(t) = m(t') * [k - d_t(b_0)] > m(t')$.

- Case $\deg_\alpha(\beta | s |) =: n \geq 1$. Then:

$$h = \mu\alpha.\beta | s | \left\{ \alpha | (\cdot) b_1 /_{\alpha | \cdot |^{(1)}}, \dots, \alpha | (\cdot) b_n /_{\alpha | \cdot |^{(n)}} \right\}$$

for a w.c. (b_1, \dots, b_n) of b_0 . So $m(t') = k - A'$ and $m(t) = k - A$, with A' and A respectively the multisets:

$$B_{t'}^c * B_{t'}^s * [d_{t'}(b) \mid b \text{ in } a v \in b_i \text{ for an } i] * [d_{t'}(b_1), \dots, d_{t'}(b_n)] \\ B_t^c * B_t^s * [d_t(b) \mid b \text{ in } a v \in b_0] * [d_t(b_0)],$$

where we put $B_t^c := [d_t(b) \mid b \text{ in } c]$ (and analogously for s, t'). Now for $i = 1, \dots, n$ we have: $d_{t'}(b_i) = d_{t'}(h) + d_h(b_i) > d_t(b_0)$ since as one sees from the expression of h , we have $d_{t'}(h) = d_t(b_0)$ and $d_h(b_i) > 0$. Also, it is easily understood that for all b occurring in c , or occurring in s , we have: $d_{t'}(b) = d_t(b)$. Finally, observe that since (b_1, \dots, b_n) is a w.c. of b_0 , then: b occurs in some $v \in b_0$ iff b occurs in some $v \in b_i$ for some i . And for all such b we have: $d_{t'}(b) = d_{t'}(v) + d_v(b) > d_t(v) + d_v(b) = d_t(b)$ since $d_{t'}(v) = d_{t'}(b_i) > d_t(b_0) = d_t(v)$. All these considerations precisely mean $m(t) > m(t')$. \square

Analogously we find:

Proposition 2.10. If $t \rightarrow_{\lambda^r} t' + \mathbb{T}$ then $m(t) > m(t')$.

However, only $m(t)$ is not enough to prove strong normalization. In fact (reasoning similarly as before):

$$\begin{aligned}
x\langle [v]/x \rangle &= v & y\langle 1/x \rangle &= y \quad (y \neq x) & (\lambda y.t)\langle [\vec{u}]/x \rangle &= \lambda y.t\langle [\vec{u}]/x \rangle \\
x\langle 1/x \rangle &= x\langle [v, w, \vec{u}]/x \rangle = 0 & y\langle [v, \vec{u}]/x \rangle &= 0 \quad (y \neq x) & (\mu\alpha.\beta|t)\langle [\vec{u}]/x \rangle &= \mu\alpha.\beta|t\langle [\vec{u}]/x \rangle \\
(t[v_1, \dots, v_n])\langle [\vec{u}]/x \rangle &= \sum_{([\vec{s}^0], \dots, [\vec{s}^n]) \text{ w.c. of } [\vec{u}]} t\langle [\vec{s}^0]/x \rangle [v_1\langle [\vec{s}^1]/x \rangle, \dots, v_n\langle [\vec{s}^n]/x \rangle].
\end{aligned}$$

Figure 1. Definition of linear substitution

$$\begin{aligned}
\langle x \rangle_\alpha [v, \vec{u}] &= 0 & \langle x \rangle_\alpha 1 &= x & \langle \eta | t \rangle_\alpha [\vec{u}] &= \eta | \langle t \rangle_\alpha [\vec{u}] \quad (\text{if } \eta \neq \alpha) \\
\langle \mu\gamma.\eta | t \rangle_\alpha [\vec{u}] &= \mu\gamma.\langle \eta | t \rangle_\alpha [\vec{u}] & \langle \lambda y.t \rangle_\alpha [\vec{u}] &= \lambda y.\langle t \rangle_\alpha [\vec{u}] & \langle \alpha | t \rangle_\alpha [\vec{u}] &= \sum_{([\vec{w}^1], [\vec{w}^2]) \text{ w.c. of } [\vec{u}]} \alpha | (\langle t \rangle_\alpha [\vec{w}^1]) [\vec{w}^2] \\
\langle t[v_1, \dots, v_n] \rangle_\alpha [\vec{u}] &= \sum_{([\vec{w}^0], \dots, [\vec{w}^n]) \text{ w.c. of } [\vec{u}]} (\langle t \rangle_\alpha [\vec{w}^0]) [\langle v_1 \rangle_\alpha [\vec{w}^1], \dots, \langle v_n \rangle_\alpha [\vec{w}^n]].
\end{aligned}$$

Figure 2. Definition of linear named application

Proposition 2.11. *If $t \rightarrow_{\rho^r} t' + \mathbb{T}$ then $m(t) \geq m(t')$, and there are cases in which the equality holds, such as (for $\beta \neq \eta$): $m(\mu\gamma.\alpha|\mu\beta.\eta|x|) = 1 = m(\mu\gamma.\eta|x|)$ with $\mu\gamma.\alpha|\mu\beta.\eta|x| \rightarrow_{\rho^r} \mu\gamma.\eta|x|$.*

That is why, in order to get a strongly normalising measure, we add another component:

Definition 2.12. *We define the measure:*

$$\tilde{m}(t) := (m(t), \deg_\mu(t)) \in !\mathbb{N} \times \mathbb{N}$$

ordered by the (well-founded) lexicographic order.

Corollary 2.13 (SN). *If $t \rightarrow_r t' + \mathbb{T}$ then $\tilde{m}(t) > \tilde{m}(t')$. Therefore, the resource reduction \rightarrow_r on sums is strongly normalising.*

Proof. The only case in which $m(\cdot)$ may remain constant is along a ρ^r -reduction, but in this case $\deg_\mu(t)$ strictly decreases. \square

Before turning to the confluence, let us see some properties of the measure $m(\cdot)$ that we will use in the following.

Lemma 2.14. *Let $c = c(\xi)$ be a single-hole context and t a $\lambda\mu$ -term. Then: $m(c(\xi)) \geq m(t)$.*

Proof. We have $m(c(\xi)) = A * [\deg_\mu(c(\xi)) - d_{c(\xi)}(b) \mid b \text{ in } t]$ where $A := [\deg_\mu(c(\xi)) - d_{c(\xi)}(b) \mid b \text{ in } c]$. But $\deg_\mu(c(\xi)) = \deg_\mu(c) + \deg_\mu(t)$ and, for all occurrence b in t , we have: $d_{c(\xi)}(b) = d_t(b) + d_c(\xi) \leq d_t(b) + \deg_\mu(c)$. Thus, for all occurrence b of bag in t , we have: $\deg_\mu(c(\xi)) - d_{c(\xi)}(b) \geq \deg_\mu(t) - d_t(b)$ and this last integer is exactly a generic element of $m(t)$ (if it is non-empty). Hence $m(c(\xi)) \geq A * m(t) \geq m(t)$. \square

However, there are cases in which $m(c(\xi)) = m(t)$ even if $c \neq \xi$. For example, taking $c = \lambda x.\xi$ one has $m(c(\xi)) = 1 = m(t)$ for all $t \in \lambda\mu^r$ not containing any bags. This is exactly

why, in the following, we will consider a slightly different size, called **ms** (defined in Corollary 2.16).

Lemma 2.15. *Let $c = c(\xi)$ be a single-hole resource context and $t, s \in \lambda\mu$. Then:*

1. $m(c(\xi)) = (\deg_\mu(t) + m(c)) * ((\deg_\mu(c) - d_c(\xi) + m(t)))$.
2. *If $\deg_\mu(s) \leq \deg_\mu(t)$ and $m(s) < m(t)$, then $m(c(s)) < m(c(t))$.*

Proof sketch. Easily checked, thanks to the clear fact that if b is the occurrence of a bag in c , then $d_{c(\xi)}(b) = d_c(b)$. \square

In the following, we will need a strong normalising measure which, in addition, satisfies the properties of the following Corollary 2.16. However, we have seen with some lines above that $\tilde{m}(\cdot)$ is not adapted for that. This is why we operate a last slight modification. First, let us consider the size $\mathbf{sz}(t) \in \mathbb{N}_{\geq 1}$ of resource $\lambda\mu$ -terms: $\mathbf{sz}(x) := 1$, $\mathbf{sz}(\lambda x.t) := 1 + \mathbf{sz}(t) := \mathbf{sz}(\mu\alpha.\beta|s|)$, $\mathbf{sz}(t_0[t_1, \dots, t_k]) := 1 + k + \sum_{i=0}^k \mathbf{sz}(t_i)$. Of course $\mathbf{sz}(t) = 1$ iff t is a variable, and for all c single-hole context, $\mathbf{sz}(c(\xi)) \geq \mathbf{sz}(t)$ where the equality holds iff $c = \xi$.

Corollary 2.16. *Define a measure $\mathbf{ms}(\cdot)$ of $\lambda\mu$ -terms as:*

$$\mathbf{ms}(t) := (\tilde{m}(t), \mathbf{sz}(t)) \in !\mathbb{N} \times \mathbb{N} \times \mathbb{N}$$

ordered lexicographically (and thus well-founded). Then:

1. t is a variable iff $\mathbf{ms}(t)$ takes its minimal value $(1, 0, 1)$.
2. For all single-hole context $c = c(\xi)$, we have $\mathbf{ms}(c(\xi)) \geq \mathbf{ms}(t)$, and the equality holds iff $c = \xi$.
3. If $t \rightarrow_r t' + \mathbb{T}$ then $\mathbf{ms}(t) > \mathbf{ms}(t')$.

2.2 Confluence

Due to the presence of three different reductions, the confluence or our resource $\lambda\mu$ -calculus is not easy. Another difficulty is raised from the fact that we placed ourselves in a qualitative setting, that is, with idempotent sums, so that we

cannot always reduce a sum component-wise. This is why we split the problem of the confluence in two steps: first, we show that the *quantitative* resource $\lambda\mu$ -calculus (that is, where sum is *not* idempotent, and thus coefficients matter) is confluent (Section 2.2.1); second, we show that its confluence implies the confluence of the calculus with no coefficients (Section 2.2.2). Before all that, let us precisely explain the notion of quantitative resource calculus:

Definition 2.17. The quantitative resource $\lambda\mu$ -calculus $\mathbb{N}\langle\lambda\mu^r\rangle$ is built as the qualitative one ($2\langle\lambda\mu^r\rangle$, Definition 2.2) except for taking now “+” non-idempotent. We define the three base-case reductions $\rightarrow_{\lambda^r}^+$, $\rightarrow_{\mu^r}^+$, $\rightarrow_{\rho^r}^+$ in $\lambda\mu^r \times \mathbb{N}\langle\lambda\mu^r\rangle$: the reduction $\rightarrow_{\rho^r}^+$ is defined as usual, while $\rightarrow_{\lambda^r}^+$ and $\rightarrow_{\mu^r}^+$ are defined as in Definition 2.5, except for the fact that the linear substitution and linear named application are replaced with a modified version of them, denoted respectively $t\langle[\vec{u}]/x\rangle^+$ and $\langle t\rangle_{\alpha}^+[\vec{u}]$, and defined in the next Definition 2.19. The contextual union of the base-reductions $\rightarrow_{\lambda^r}^+$, $\rightarrow_{\mu^r}^+$, $\rightarrow_{\rho^r}^+$ forms a reduction $\rightarrow_{\mathbb{r}}^+$ on $\lambda\mu^r \times \mathbb{N}\langle\lambda\mu^r\rangle$ which is extended to all $\mathbb{N}\langle\lambda\mu^r\rangle \times \mathbb{N}\langle\lambda\mu^r\rangle$ by taking $\{(t + \mathbb{S}, \mathbb{T} + \mathbb{S}) \mid t \rightarrow_{\mathbb{r}}^+ \mathbb{T}\}$ (remark that we dropped the annoying condition “ $t \notin \mathbb{S}$ ”, since now coefficients matter; it is the main reason why we turn to this calculus).

Notation 2.18. If $[u_1, \dots, u_k]$ is a bag – with the written enumeration of (possibly multiple) elements – and W is a function $W : \{1, \dots, k\} \rightarrow I = \{i_0 < \dots < i_n\}$, we will sometimes denote it by $W : (u_1, \dots, u_k) \rightarrow I$, or by $W : (\vec{u}) \rightarrow I$. When we use such notation we mean that W generates the w.c. $([u_j \mid j \in W^{-1}(i_0)], \dots, [u_j \mid j \in W^{-1}(i_n)])$ of $[u_1, \dots, u_k]$, and denoted by $([\vec{w}^{i_0}], \dots, [\vec{w}^{i_n}])$. In the case $[\vec{u}] = 1$, we write $W : () \rightarrow I$ and we say that there is exactly one w.c. generated by W , namely $(1, (n+1 \text{ times}), 1)$.

Definition 2.19. The quantitative version $t\langle[\vec{u}]/x\rangle^+$ of the linear substitution is defined exactly as in Figure 1 but by replacing the sum on all the $([\vec{w}^0], \dots, [\vec{w}^n])$ w.c. of $[\vec{u}]$ with the sum on all $W : (\vec{u}) \rightarrow \{0, \dots, n\}$, and by taking the above w.c.’s as the ones generated by W . The quantitative version $\langle t\rangle_{\alpha}^+[\vec{u}]$ of the linear named application is defined exactly as in Figure 2 but by replacing, in the case of an application, the sum on all the $([\vec{w}^0], \dots, [\vec{w}^n])$ w.c. of $[\vec{u}]$ with the sum on all $W : (\vec{u}) \rightarrow \{0, \dots, n\}$, and by taking the above w.c.’s as the ones generated by W . Analogously for the case of a named term, where we use $W : (\vec{u}) \rightarrow \{1, 2\}$.

For instance: $(\mu\alpha.\alpha|\mu\eta.\alpha|x|)|[y, y] \rightarrow_{\mathbb{r}}^+ \mu\alpha.\alpha|(\mu\eta.\alpha|x1|)[y, y] + 2\mu\alpha.\alpha|(\mu\eta.\alpha|x|y|)|[y]| + \mu\alpha.\alpha|(\mu\eta.\alpha|x|y, y|)|1|$.

In the following, $\text{supp}(\mathbb{T}) \in 2\langle\lambda\mu^r\rangle$ is the *support* of a $\mathbb{T} \in \mathbb{N}\langle\lambda\mu^r\rangle$, that is, the set of its addends (with no coefficients): $\text{supp}(\mathbb{T})$ is \mathbb{T} when considered with an idempotent “+”.

Remark 2.20. It is clear by the definitions that if, for $t \in \lambda\mu^r$, one has $t \rightarrow_{\mathbb{r}} \mathbb{T}$ (in $2\langle\lambda\mu^r\rangle$) and $t \rightarrow_{\mathbb{r}}^+ \mathbb{S}$ (in $\mathbb{N}\langle\lambda\mu^r\rangle$) by reducing the same redex, then $\text{supp}(\mathbb{S}) = \mathbb{T}$. That is, the two reductions only differ for the coefficients. Said differently,

the qualitative substitutions $t\langle[\vec{u}]/x\rangle$ and $\langle t\rangle_{\alpha}[\vec{u}]$ are just the quantitative substitutions $t\langle[\vec{u}]/x\rangle^+$ and $\langle t\rangle_{\alpha}^+[\vec{u}]$ taken with boolean coefficients.

Remark 2.21. Using the fact that the reduction $\rightarrow_{\mathbb{r}}$ is strongly normalising in $\lambda\mu^r$ (Corollary 2.13), we can prove that the reduction $\rightarrow_{\mathbb{r}}^+$ is strongly normalising in $\mathbb{N}\langle\lambda\mu^r\rangle$. It suffices to extend the strongly normalising measure $\tilde{m}(\cdot)$ of $\lambda\mu^r$ (Definition 2.12) to $\mathbb{N}\langle\lambda\mu^r\rangle$ by setting $\tilde{m}(\mathbb{T}) := [\tilde{m}(t) \mid t \in \mathbb{T}] \in !(\mathbb{N} \times \mathbb{N})$, and use the multiset order.

Remark 2.22 (Embedding inside the differential $\lambda\mu$ -calculus). In [29], Vaux defines a differential $\lambda\mu$ -calculus, let us call it $(\lambda\mu^{\partial}, \rightarrow_{\partial})$ in this remark, and proves its confluence. Our resource $\lambda\mu$ -calculus $2\langle\lambda\mu^r\rangle$ and $\mathbb{N}\langle\lambda\mu^r\rangle$ are strictly related to it, as they translate into $\lambda\mu^{\partial}$ via⁴ $(\cdot)^{\partial} : \lambda\mu^r \rightarrow \lambda\mu^{\partial}$ defined as: $x^{\partial} := x$, $(\lambda x.t)^{\partial} := \lambda x.t^{\partial}$, $(\mu\alpha.\beta|t|)^{\partial} := \mu\alpha.\beta|t^{\partial}|$, $(t[u_1, \dots, u_k])^{\partial} := (\mathbf{d}^k t^{\partial} \bullet (u_1^{\partial}, \dots, u_k^{\partial})) \mathbf{0}$. We can extend it to sums, both in $2\langle\lambda\mu^r\rangle$ and in $\mathbb{N}\langle\lambda\mu^r\rangle$, by linearity. In the qualitative case (that is, if we consider $(\cdot)^{\partial} : 2\langle\lambda\mu^r\rangle \rightarrow \lambda\mu^{\partial}$), it is not a well-behaved embedding, because it does not preserve reductions. On the contrary, it does in the quantitative case (that is, if we consider $(\cdot)^{\partial} : \mathbb{N}\langle\lambda\mu^r\rangle \rightarrow \lambda\mu^{\partial}$), in the sense that: if $t \rightarrow_{\lambda\mu^r}^+ \mathbb{T}$ in $\mathbb{N}\langle\lambda\mu^r\rangle$, then $t^{\partial} \rightarrow_{\partial} \mathbb{T}^{\partial}$ in $\lambda\mu^{\partial}$ (“ \rightarrow ” is the reflexive transitive closure of \rightarrow).

One may wonder if it is possible to use the confluence of $(\lambda\mu^{\partial}, \rightarrow_{\partial})$ to infer the confluence of our calculi. In fact, it is possible to show that the local confluence of $\rightarrow_{\lambda\mu^r}^+$ follows from the confluence of \rightarrow_{∂} . However, as the reader has probably noticed, we only talked about $\rightarrow_{\lambda\mu^r}^+$, and not about the whole $\rightarrow_{\mathbb{r}}^+ = \rightarrow_{\lambda\mu^r}^+ \cup \rightarrow_{\rho^r}^+$. This is simply because in [29] the ρ -reduction is not considered. Remark that, even if it is possible to prove the confluence of \rightarrow_{ρ^r} by itself, we cannot use it in order to entail the confluence of $\rightarrow_{\mathbb{r}}^+ = \rightarrow_{\lambda\mu^r}^+ \cup \rightarrow_{\rho^r}^+$ by invoking the well-known Hindley-Rosen lemma. This is because $\rightarrow_{\rho^r}^+$ and $\rightarrow_{\lambda\mu^r}^+$ do not commute, as the following example shows (where $\gamma \neq \eta \neq \alpha$): $\mu\alpha.\alpha|(\mu\gamma.\eta|x|)1| \xrightarrow{\mu^+} \leftarrow (\mu\alpha.\alpha|\mu\gamma.\eta|x|)1 \xrightarrow{\rho^+} (\mu\alpha.\eta|x|)1 \xrightarrow{\rho^+} \mu\alpha.\eta|x|$, but $\mu\alpha.\alpha|(\mu\gamma.\eta|x|)1| \not\xrightarrow{\rho^+} \mu\alpha.\eta|x|$. In the previous “non-reduction”, the blocked ρ -redex can be unblocked by performing a μ -reduction (and the diagram closes). O. Laurent suggests (private communication) that we could still use the confluence of $(\lambda\mu^{\partial}, \rightarrow_{\partial})$ in order to obtain the confluence of $\rightarrow_{\mathbb{r}}^+ = \rightarrow_{\lambda\mu^r}^+ \cup \rightarrow_{\rho^r}^+$ passing through a factorization lemma: if $t \rightarrow_{\mathbb{r}}^+ \mathbb{T}$ then $t \xrightarrow{\mu^+} \lambda\mu^r \mathbb{T}' \xrightarrow{\rho^+} \mathbb{T}'$, for some \mathbb{T}' .

2.2.1 Confluence of $(\mathbb{N}\langle\lambda\mu^r\rangle, \rightarrow_{\mathbb{r}}^+)$. We present here a proof which essentially consists in closing the diagrams of all the possible critical pairs.

⁴Here we are considering that the reader knows the syntax of $\lambda\mu^{\partial}$.

Remark 2.23. We can extend the definition of linear substitution and linear named application to sums by linearity. Analogously, the renaming of a sum $\mathbb{T}\{\alpha/\beta\}$ is defined component-wise. With these definitions in place one checks that base-step-reduction lifts to sums, i.e. $(\mu\alpha.\beta|\mathbb{T}|) [\vec{U}] \rightarrow_{\mu^r}^+ \mu\alpha.\langle\mathbb{T}\rangle_{\alpha}^+ [\vec{U}]$ and analogously for $(\lambda x.\mathbb{T}) [\vec{U}]$ and $\mu\alpha.\beta|\mu\gamma.\eta|\mathbb{T}|$. One can also check that \rightarrow_{τ}^+ on $\mathbb{N}\langle\lambda\mu^r\rangle$ is contextual.

Notation 2.24. In this section we will sometimes use the following notation: for α, β, η names, we set $\delta_{\eta}^{\alpha}(\beta)$ to be α if $\beta = \eta$, or η otherwise.

The following is the crucial technical lemma.

Lemma 2.25. Let $t, s \in \lambda\mu^r$, x a variable, α, β names and $[\vec{u}]$ a bag. If $s \rightarrow_{\tau}^+ \mathbb{S}$ then:

1. $s\{\alpha/\beta\} \rightarrow_{\tau}^+ \mathbb{S}\{\alpha/\beta\}$
2. $t\langle[s, \vec{u}]/x\rangle^+ \rightarrow_{\tau}^+ t\langle[\mathbb{S}, \vec{u}]/x\rangle^+$
3. $s\langle[\vec{u}]/x\rangle^+ \rightarrow_{\tau}^+ \mathbb{S}\langle[\vec{u}]/x\rangle^+$
4. $\langle t \rangle_{\alpha}^+ [s, \vec{u}] \rightarrow_{\tau}^+ \langle t \rangle_{\alpha}^+ [\mathbb{S}, \vec{u}]$
5. $\mu\alpha.\langle\beta|t\rangle_{\alpha}^+ [s, \vec{u}] \rightarrow_{\tau}^+ \mu\alpha.\langle\beta|t\rangle_{\alpha}^+ [\mathbb{S}, \vec{u}]$
6. $\langle s \rangle_{\alpha}^+ [\vec{u}] \rightarrow_{\tau}^+ \langle \mathbb{S} \rangle_{\alpha}^+ [\vec{u}]$.
7. $\mu\alpha.\langle\beta|s\rangle_{\alpha}^+ [\vec{u}] \rightarrow_{\tau}^+ \mu\alpha.\langle\beta|\mathbb{S}\rangle_{\alpha}^+ [\vec{u}]$.

Before proving it, let us remark that, in the qualitative setting, it is false. For instance, if $s \rightarrow_{\tau} s'$, then $(x[x])\langle[s, s]/x\rangle = s[s] \not\rightarrow_{\tau} s[s'] + s'[s] = (x[x])\langle[s, s']/x\rangle$. In the quantitative case, instead, $(x[x])\langle[s, s]/x\rangle^+ = 2s[s] \rightarrow_{\tau}^+ s[s'] + s'[s] = (x[x])\langle[s, s']/x\rangle^+$.

Proof sketch of Lemma 2.25. 1). Induction on s . The only interesting cases are:

- Case $s = \mu\gamma.\eta|s'|$: we have two subcases: Subcase $s \rightarrow_{\tau}^+ \mathbb{S}$ is performed by reducing s' : easy by inductive hypothesis. Subcase $s' = \mu\gamma'.\eta'|s''$ and $s \rightarrow_{\tau}^+ \mathbb{S}$ is performed by reducing its leftmost ρ -redex: then $s = \mu\gamma.\eta|\mu\gamma'.\eta'|s''|$, $\mathbb{S} = \mu\gamma.\eta'|s''|\{\eta/\gamma'\}$ and we have the four sub-subcases $\eta = \beta$ and $\eta' = \beta$, or $\eta = \beta$ and $\eta' \neq \beta$, or $\eta \neq \beta$ and $\eta' = \beta$, or $\eta \neq \beta$ and $\eta' \neq \beta$. They are all similar, let us only show the second one, for which we have:

$$\begin{aligned} \mathbb{S}\{\alpha/\beta\} &= \mu\gamma.\eta'|s''|\{\alpha/\beta, \alpha/\gamma'\} = \mu\gamma.\delta_{\eta'}^{\alpha}(\gamma')|s''\{\alpha/\beta, \alpha/\gamma'\} \\ s\{\alpha/\beta\} &= \mu\gamma.\alpha|\mu\gamma'.\eta'|s''\{\alpha/\beta\} \rightarrow_{\rho}^+ \mu\gamma.\eta'|s''\{\alpha/\beta\}|\{\alpha/\gamma'\} \\ &= \mu\gamma.\delta_{\eta'}^{\alpha}(\gamma')|s''\{\alpha/\beta, \alpha/\gamma'\} = \mathbb{S}\{\alpha/\beta\}. \end{aligned}$$

- Case $s = s'[\vec{v}]$: we have four subcases depending on how the reduction $s \rightarrow_{\tau}^+ \mathbb{S}$ is performed. The only interesting one is the subcase $s' = \mu\gamma.\eta|s''$ and $s \rightarrow_{\tau}^+ \mathbb{S}$ is performed by reducing the μ -redex s , for which we have: $\mathbb{S} = \mu\gamma.\langle\eta|s''\rangle_{\gamma}^+ [\vec{v}]$ and $s\{\alpha/\beta\} = (\mu\gamma.\delta_{\eta}^{\alpha}(\beta)|s''\{\alpha/\beta\})[\vec{v}\{\alpha/\beta\}] \rightarrow_{\mu^r}$ -reduces to $\mu\gamma.\langle\delta_{\eta}^{\alpha}(\beta)|s''\{\alpha/\beta\}\rangle_{\gamma}^+ [\vec{v}\{\alpha/\beta\}]$ which in turn coincides with the sum $\mu\gamma.\langle\eta|s''|\{\alpha/\beta\}\rangle_{\gamma}^+ [\vec{v}\{\alpha/\beta\}] = \mathbb{S}\{\alpha/\beta\}$.

(2). Induction on t . The only non-trivial case is when t is $v_0[v_1, \dots, v_n]$. In this case we can write $t\langle[s, \vec{u}]/x\rangle^+$ as:

$$\sum_W \sum_{j=0}^n (v_0\langle[\vec{w}^0] * [s]_0^j/x\rangle) [\dots, v_i\langle[\vec{w}^i] * [s]_i^j/x\rangle, \dots]$$

where $W : (\vec{u}) \rightarrow \{1, \dots, n\}$ and we put $[s]_i^j$ to be the singleton multiset $[s]$ if $i = j$, and the empty multiset 1 if $i \neq j$. Fix now a $W : (\vec{u}) \rightarrow \{1, \dots, n\}$ (together with its generated w.c.) and consider each of the $n + 1$ elements of the sum on j . We write the case for $j = 0$, but the other cases are exactly the same. Since $j = 0$, the element is $(v_0\langle[\vec{w}^0] * [s]/x\rangle) [\dots, v_i\langle[\vec{w}^i]/x\rangle, \dots]$ and by inductive hypothesis it \rightarrow_{τ}^+ -reduces to $(v_0\langle[\vec{w}^0] * [\mathbb{S}]/x\rangle) [\dots, v_i\langle[\vec{w}^i]/x\rangle, \dots]$. Now summing up all the elements for $j = 0, \dots, n$ and $W : (\vec{u}) \rightarrow \{1, \dots, n\}$ we obtain the following sum:

$$\sum_W \sum_{j=0}^n (v_0\langle[\vec{w}^0] * [\mathbb{S}]_0^j/x\rangle) [\dots, v_i\langle[\vec{w}^i] * [\mathbb{S}]_i^j/x\rangle, \dots].$$

which can be shown to be the desired $(v_0[v_1, \dots, v_n]) \langle[\mathbb{S}, \vec{u}]/x\rangle^+$.

(3). Induction on s . We only show the case $s = \mu\alpha.\beta|s'|$, which splits in two subcases: the subcase where $s \rightarrow_{\tau}^+ \mathbb{S}$ is performed by reducing s' is immediate. The subcase where $s' = \mu\gamma.\eta|s''$ and $s \rightarrow_{\tau}^+ \mathbb{S}$ is performed by reducing its leftmost ρ -redex goes as follows: we have $\mathbb{S} = \mu\alpha.\eta|s''|\{\beta/\gamma\}$ and

$$\begin{aligned} s\langle[\vec{u}]/x\rangle^+ &= \mu\alpha.\beta|\mu\gamma.\eta|s''\langle[\vec{u}]/x\rangle^+ \\ &\rightarrow_{\rho^r} \mu\alpha.\eta|s''\langle[\vec{u}]/x\rangle^+|\{\beta/\gamma\} = \mu\alpha.\eta|s''\langle[\vec{u}]/x\rangle^+|\{\beta/\gamma\} \\ &= \mu\alpha.\eta|s''|\{\beta/\gamma\}\langle[\vec{u}]/x\rangle^+ = \mathbb{S}\langle[\vec{u}]/x\rangle^+. \end{aligned}$$

(4). Induction on t . Similar to point (2).

(5). It is easy discriminating the cases $\alpha = \beta$ and $\alpha \neq \beta$ and concluding by point (4).

(6). Induction on $s \in \lambda\mu$. The only interesting cases are:

- Case $s = \mu\beta.\gamma|s'|$. We have two subcases: the subcase where $s \rightarrow_{\tau}^+ \mathbb{S}$ is performed by reducing s' , so $\mathbb{S} = \mu\beta.\gamma|s'|$ with $s' \rightarrow_{\tau} \mathbb{S}'$, is easy by inductive hypothesis (however remark that we *cannot* immediately apply the inductive hypothesis on $\gamma|s'|$, simply because the named term $\gamma|s'| \notin \lambda\mu^r$). The subcase where $s' = \mu\gamma'.\eta'|s''$ (with $\gamma \neq \gamma'$) and $s \rightarrow_{\tau}^+ \mathbb{S}$ is performed by reducing its leftmost ρ -redex goes as follows: we have $\mathbb{S} = \mu\beta.\eta|s''|\{\gamma/\gamma'\}$ and we split in two sub-subcases depending whether $\alpha \neq \gamma$ or $\alpha = \gamma$. Let us only show this last sub-subcase: We have (putting $W : (\vec{u}) \rightarrow \{1, 2\}$):

$$\begin{aligned} \langle s \rangle_{\alpha}^+ [\vec{u}] &= \sum_W \mu\beta.\alpha | (\mu\gamma'.\langle\eta|s''\rangle_{\alpha}^+ [\vec{w}^1]) [\vec{w}^2] \\ &\rightarrow_{\mu^r}^+ \sum_W \mu\beta.\alpha |\mu\gamma'.\langle\langle\eta|s''\rangle_{\alpha}^+ [\vec{w}^1]\rangle_{\gamma'}^+ [\vec{w}^2] \\ &\rightarrow_{\rho^r}^+ \sum_W \mu\beta.\langle\langle\eta|s''\rangle_{\alpha}^+ [\vec{w}^1]\rangle_{\gamma'}^+ [\vec{w}^2] \{\alpha/\gamma'\} \\ &= \sum_W \langle\langle\mu\beta.\eta|s''\rangle_{\alpha}^+ [\vec{w}^1]\rangle_{\gamma'}^+ [\vec{w}^2] \{\alpha/\gamma'\} \\ &= \langle\mu\beta.\eta|s''|\{\alpha/\gamma'\}\rangle_{\alpha}^+ [\vec{u}] \\ &= \langle\mathbb{S}\rangle_{\alpha}^+ [\vec{u}]. \end{aligned}$$

- Case $s = s'[v_1, \dots, v_n]$: we have four subcases depending on how the reduction $s \rightarrow_{\tau}^+ \mathbb{S}$ is performed. We only show the one in which $s' = \mu\gamma.\eta|s''$ (with $\gamma \neq \alpha$) and $s \rightarrow_{\tau}^+ \mathbb{S}$ is performed by reducing the μ -redex s . In this subcase we have

$\mathbb{S} = \mu\gamma.\langle\eta|s''|\rangle_Y^+[\vec{v}]$ and (putting $W : (\vec{u}) \longrightarrow \{0, \dots, n\}$):

$$\begin{aligned} \langle s \rangle_\alpha^+[\vec{u}] &= \sum_W (\langle \mu\gamma.\eta|s''|\rangle_\alpha^+[\vec{w}^0]) [\dots, \langle v_i \rangle_\alpha^+[\vec{w}^i], \dots] \\ &\xrightarrow{\mu^+} \mu\gamma.\sum_W (\langle \eta|s''|\rangle_\alpha^+[\vec{w}^0]) [\dots, \langle v_i \rangle_\alpha^+[\vec{w}^i], \dots] \\ &= \mu\gamma.\langle\eta|s''|\rangle_Y^+[\vec{v}]_\alpha^+[\vec{u}] \\ &= \langle \mathbb{S} \rangle_\alpha^+[\vec{u}]. \end{aligned}$$

(7). It is immediate by discriminating the cases $\alpha = \beta$ and $\alpha \neq \beta$ and then concluding by point (6). \square

Proposition 2.26. *The reduction \rightarrow_r^+ is locally confluent in $\mathbb{N}\langle\lambda\mu^r\rangle$.*

Proof sketch. We show, by induction on a single-hole resource context c , that if $t \rightarrow_{\text{base}^r}^+ \mathbb{T}$ and $c(t) \rightarrow_r^+ \mathbb{T}_2$, then there is $\mathbb{T}' \in \mathbb{N}\langle\lambda\mu^r\rangle$ s.t. $c(\mathbb{T}) \rightarrow_r^+ \mathbb{T}' \xleftarrow{r} \mathbb{T}_2$. The proof crucially uses Lemma 2.25 and Remark 2.23. All the cases of the induction are either easy by induction, or they reduce to the case $c = \xi$, so this is the only one we sketch below.

We have $c(t) = t \rightarrow_{\text{base}^r}^+ \mathbb{T}$ and we only have the three base-cases of Definition 2.17.

Case $t = (\lambda x.s)[\vec{u}]$ and $\mathbb{T} = s\langle\vec{u}\rangle/x^+$. Then $c(t) = t \rightarrow_r^+ \mathbb{T}_2$ (on a different redex than t) can only be performed either by reducing s , or by reducing an element w of $[\vec{u}]$. We have thus the two easy respective diagrams.

Case $t = (\mu\alpha.\beta|s|)[\vec{u}]$ and $\mathbb{T} = \mu\alpha.\langle\beta|s|\rangle_\alpha^+[\vec{u}]$. Then $c(t) = t \rightarrow_r^+ \mathbb{T}_2$ (on a different redex than t) can only be performed either by reducing s , giving rise to an easy diagram, or by reducing an element w of $[\vec{u}]$, giving rise to an easy diagram, or if $s = \mu\gamma.\eta|s'|$ and we reduce the ρ -redex $\dots\beta|\mu\gamma.\dots|$. In the latter case we split into the case $\alpha \neq \beta$, the case $\alpha = \beta, \gamma \neq \eta, \eta = \alpha$, the case $\alpha = \beta, \gamma \neq \eta, \eta \neq \alpha$, and the case $\alpha = \beta, \eta = \gamma$ (with necessarily $\gamma \neq \alpha$). These four cases respectively correspond to four non-trivial (but similar) diagrams, of which we only show the one corresponding to the case $\alpha = \beta, \gamma \neq \eta, \eta = \alpha$: $(\mu\alpha.\alpha|\mu\gamma.\alpha|s'|)[\vec{u}]$ reduces both to $\mathbb{U} := \mu\alpha.\langle\alpha|\mu\gamma.\alpha|s'| \rangle_\alpha^+[\vec{u}]$ and to $v := (\mu\alpha.\alpha|s'\{\alpha/\gamma\})[\vec{u}]$. Now, $v \rightarrow_r^+ \mu\alpha.\langle\alpha|s'\{\alpha/\gamma\} \rangle_\alpha^+[\vec{u}] = \sum_W \mu\alpha.\alpha|(\langle s'\{\alpha/\gamma\} \rangle_\alpha^+[\vec{w}^0])[\vec{w}^1]$

$=: \mathbb{V}$ (with $W : (\vec{u}) \rightarrow \{1, 2\}$), while it is easy to see that (with $W : (\vec{u}) \rightarrow \{1, 2\}, D : (\vec{w}^0) \rightarrow \{1, 2\}$) $\mathbb{U} \rightarrow_r^+$ -reduces to $\sum_W \sum_D \mu\alpha.\alpha|\mu\gamma.\alpha|(\langle s' \rangle_\alpha^+[\vec{d}^0])[\vec{d}^1]_\gamma^+[\vec{w}^1]$ which in turn \rightarrow_r^+ -reduces to $\sum_W \sum_D \mu\alpha.\alpha|(\langle s' \rangle_\alpha^+[\vec{d}^0])[\vec{d}^1]_\gamma^+[\vec{w}^1]|\{\alpha/\gamma\} =: \mathbb{U}'$.

We can show that $\mathbb{V} = \mathbb{U}'$, so the diagram is closed.

Case $t = \mu\gamma.\alpha|\mu\beta.\eta|s|$ and $\mathbb{T} = \mu\gamma.\eta|s|\{\alpha/\beta\}$. Then $c(t) = t \rightarrow_r^+ \mathbb{T}_2$ (on a different redex than t) can be only performed either by reducing s , which gives an easy diagram, or if $s = \mu\gamma'.\eta'|s'|$ and we reduce the ρ -redex $\dots\eta|\mu\gamma'.$. Putting $\delta_0 := \delta_\eta^\alpha(\beta)$, $\delta'_1 := \delta_{\eta'}^\alpha(\beta)$, $\delta_1 := \delta_{\delta'_1}^{\delta_0}(\gamma')$, $\delta'_2 := \delta_{\eta'}^\eta(\gamma')$ and $\delta_2 := \delta_{\delta'_2}^\alpha(\beta)$, the latter case gives the following diagram: $\mu\gamma.\alpha|\mu\beta.\eta|\mu\gamma'.\eta'|s'|$ reduces both to $\mathbb{U} := \mu\gamma.\delta_0|\mu\gamma'.\delta'_1|s'\{\alpha/\beta\}$ and to $\mathbb{V} := \mu\gamma.\alpha|\mu\beta.\delta'_2|s'\{\eta/\gamma'\}$, while $\mathbb{U} \rightarrow_r^+$ -reduces to $\mu\gamma.\delta_1|s'\{\alpha/\beta\}\{\delta_0/\gamma'\}$ and $\mathbb{V} \rightarrow_r^+$ -reduces to $\mu\gamma.\delta_2|s'\{\eta/\gamma'\}\{\alpha/\beta\}$. We can show those sums equal, so the diagram is closed. \square

Corollary 2.27. *The reduction \rightarrow_r^+ is confluent in $\mathbb{N}\langle\lambda\mu^r\rangle$.*

Proof. By the well-known Newman Lemma, thanks to Remark 2.21 and Proposition 2.26. \square

2.2.2 From the confluence of $(\mathbb{N}\langle\lambda\mu^r\rangle, \rightarrow_r^+)$ to the confluence of $(2\langle\lambda\mu^r\rangle, \rightarrow_r)$.

Definition 2.28. *The reduction $\Rightarrow \subseteq \mathbb{N}\langle\lambda\mu^r\rangle \times \mathbb{N}\langle\lambda\mu^r\rangle$ is defined as the contextual closure of the relation:*

$$\{(m t + \mathbb{S}, m \mathbb{T} + \mathbb{S}) \mid m \in \mathbb{N}, t \rightarrow_r^+ \mathbb{T}, t \notin \text{supp}(\mathbb{S})\}.$$

We have $\Rightarrow \subseteq \rightarrow_r^+$. It is also easily seen that if $\mathbb{T} \rightarrow_r \mathbb{S}$ in $2\langle\lambda\mu^r\rangle$, then for all $m_t \in \mathbb{N}$ (with $t \in \mathbb{T}$), we have $\sum_{t \in \mathbb{T}} m_t t \Rightarrow \mathbb{S}'$ (in $\mathbb{N}\langle\lambda\mu^r\rangle$), with $\text{supp}(\mathbb{S}') = \mathbb{S}$.

Corollary 2.29. *The reduction \rightarrow_r in $2\langle\lambda\mu^r\rangle$ is locally confluent.*

Proof. Let $\mathbb{T}_1 \xleftarrow{r} t \rightarrow_r \mathbb{T}_2$ in $2\langle\lambda\mu^r\rangle$. Since we know that \rightarrow_r is strongly normalising (Corollary 2.13), there are (in $2\langle\lambda\mu^r\rangle$) reductions $\mathbb{T}_1 \rightarrow_r \mathbb{S}_1$ and $\mathbb{T}_2 \rightarrow_r \mathbb{S}_2$, with $\mathbb{S}_1, \mathbb{S}_2$ r -normal. Therefore we have (in $\mathbb{N}\langle\lambda\mu^r\rangle$) reductions $t \Rightarrow \dots \Rightarrow \mathbb{S}'_1$ and $t \Rightarrow \dots \Rightarrow \mathbb{S}'_2$, for some $\mathbb{S}'_1, \mathbb{S}'_2 \in \mathbb{N}\langle\lambda\mu^r\rangle$ s.t. $\text{supp}(\mathbb{S}'_i) = \mathbb{S}_i$. But then, since \mathbb{S}_i is r -normal, \mathbb{S}'_i must be \rightarrow_r^+ -normal. Now because of Corollary 2.27, it must be $\mathbb{S}'_1 = \mathbb{S}'_2$, and therefore $\mathbb{S}_1 = \text{supp}(\mathbb{S}'_1) = \text{supp}(\mathbb{S}'_2) = \mathbb{S}_2$. Hence, we found a common reduct of $\mathbb{T}_1, \mathbb{T}_2$. \square

Corollary 2.30 (Confluence). *The reduction \rightarrow_r is confluent on $2\langle\lambda\mu^r\rangle$.*

Proof. By Newman Lemma, thanks to Corollary 2.13 and Corollary 2.29. \square

3 Qualitative Taylor expansion

3.1 Crucial properties

The calculus and its resource sensitive version are almost the same; the Taylor expansion map makes us pass from one to the other.

Definition 3.1. *The (qualitative) Taylor expansion is the map $\mathcal{T} : \lambda\mu \rightarrow \mathcal{P}(\lambda\mu^r)$ defined as:*

$$\begin{aligned} \mathcal{T}(x) &:= \{x\} & \mathcal{T}(\lambda x.M) &:= \{\lambda x.t \mid t \in \mathcal{T}(M)\} \\ \mathcal{T}(\mu\alpha.\beta|M|) &:= \{\mu\alpha.\beta|t| \mid t \in \mathcal{T}(M)\} \\ \mathcal{T}(MN) &:= \{t[\vec{u}] \mid t \in \mathcal{T}(M), [\vec{u}] \in !\mathcal{T}(N)\}. \end{aligned}$$

Since \rightarrow_r is confluent and strongly normalising, all resource terms t have a unique r -normal form $\text{nf}_r(t) \in 2\langle\lambda\mu^r\rangle$ (it can be 0). Therefore, for all $M \in \lambda\mu$ there always exists $\text{NFT}(M) := \bigcup_{t \in \mathcal{T}(M)} \text{nf}_r(t) \subseteq \lambda\mu^r$ (in general infinite, thus not a sum). This allows to endow $\lambda\mu$ with a preorder:

$$M \leq N \text{ iff } \text{NFT}(M) \subseteq \text{NFT}(N).$$

Theorem 3.2 (Monotonicity). *For C a context, the map $C(\cdot) : \lambda\mu \rightarrow \lambda\mu$ is monotone w.r.t. \leq .*

Proof. Induction on C , as in [1]. \square

The following technical lemma says that Taylor expansion behaves well w.r.t. substitutions.

Lemma 3.3. *One has:*

1. $\mathcal{T}(M\{\alpha/\beta\}) = \mathcal{T}(M)\{\alpha/\beta\}$
2. $\mathcal{T}(M\{N/x\}) = \bigcup_{t \in \mathcal{T}(M)} \bigcup_{\vec{u} \in !\mathcal{T}(N)} t\langle[\vec{u}]/x\rangle$
3. $\mathcal{T}((M)_\alpha N) = \bigcup_{t \in \mathcal{T}(M)} \bigcup_{\vec{u} \in !\mathcal{T}(N)} \langle t \rangle_\alpha [\vec{u}]$.

Proof sketch. (1). Straightforward induction on M .
 (2). Induction on M as one does for λ -calculus. The only new case is $M = \mu\beta_\alpha|P|$ but it is done straightforwardly exactly as the case $M = \lambda x.P$.
 (3). Induction on M . Not more difficult than (2). \square

The following important “simulation property” says in which sense the elements of $\mathcal{T}(M)$ approximate M .

Proposition 3.4. *If $M \rightarrow_{\text{base}} N$, then:*

1. for all $s \in \mathcal{T}(M)$ there exist $\mathbb{T} \subseteq \mathcal{T}(N)$ s.t. $s \rightarrow_r \mathbb{T}$
2. for all $s' \in \mathcal{T}(N)$ there is $s \in \mathcal{T}(M)$ s.t. $s \rightarrow_r s' + \mathbb{T}$ for some sum $\mathbb{T} \subseteq \mathcal{T}(N)$.

Furthermore, the same property lifts to all \rightarrow , that is, if $M \rightarrow N$ then point (1) and (2) hold.

Proof sketch. Points 1) and 2) are easy using Lemma 3.3. The “furthermore” part is by induction on the single-hole context C s.t. $M = C\langle M' \rangle$, $N = C\langle N' \rangle$ and $M' \rightarrow_{\text{base}} N'$. \square

The following technical lemma is an adaptation of [11, Theorem 20].

Lemma 3.5. *Let P, Q be $\lambda\mu$ -terms, $p, p' \in \mathcal{T}(P)$ and $[\vec{d}], [\vec{d}'] \in !\mathcal{T}(Q)$. Then $p = p'$ and $[\vec{d}] = [\vec{d}']$ are entailed by any of the following three⁵ conditions:*

1. if $p\langle[\vec{d}]/x\rangle \cap p'\langle[\vec{d}']/x\rangle \neq \emptyset$
2. if $\langle p \rangle_\gamma [\vec{d}] \cap \langle p' \rangle_\gamma [\vec{d}'] \neq \emptyset$
3. if $\langle \eta | p \rangle_\gamma [\vec{d}] \cap \langle \eta | p' \rangle_\gamma [\vec{d}'] \neq \emptyset$.

The following “non-interference property” (Theorem 3.6) was first proved by Ehrhard and Regnier in [11, Theorem 22] for the λ -calculus. It is known that it fails in MELL. A natural question, to which we do not have an answer yet, is what is the threshold, between λ -calculus and MELL, where this property starts failing. It is important also because somehow it is linked to the possibility of defining a *coherence* on resource terms for which Taylor expansion is a maximal clique. We show below that the result holds in $\lambda\mu$ -calculus.

Theorem 3.6. *If $t, s \in \mathcal{T}(M)$, $t \neq s$, then $\text{nf}_r(t) \cap \text{nf}_r(s) = \emptyset$.*

⁵Remark that point 3. (used in the proof of Theorem 3.6) is *not* an inductive step of point 2., simply because $\eta|p| \notin \lambda\mu^r$. Therefore we treat it separately. This is due to the fact that we are in $\lambda\mu$ -calculus and not in Saurin’s $\Lambda\mu$ -calculus.

Proof. By induction on $\mathbf{ms}(t)$ we prove that for all $s \in \lambda\mu^r$, if $t, s \in \mathcal{T}(M)$ for some $M \in \lambda\mu$, and if there is $h \in \text{nf}_r(t) \cap \text{nf}_r(s)$, then $t = s$.

Case $\mathbf{ms}(t) = (1, 0, 1)$. Then (Corollary 2.16) t is a variable, thus M is the same variable and therefore $s = t$.

Case $\mathbf{ms}(t) > (1, 0, 1)$. By Lemma 1.4, M has shape:

$$M = \lambda\vec{x}_1\mu\alpha_1.\beta_1 | \dots | \lambda\vec{x}_k\mu\alpha_k.\beta_k | RQ_1 \dots Q_n |$$

for R either a variable, or a λ -redex or a μ -redex. Since the series of λ and μ abstractions (with their respective namings) will play no role in the following, in this proof we shorten $\lambda\vec{x}_1\mu\alpha_1.\beta_1 | \dots | \lambda\vec{x}_k\mu\alpha_k.\beta_k | \dots |$ to just $\vec{\lambda}\mu | \dots |$. So: $t = \vec{\lambda}\mu | t' [\vec{u}^1] \dots [\vec{u}^n] |$ and $s = \vec{\lambda}\mu | s' [\vec{v}^1] \dots [\vec{v}^n] |$ for $t', s' \in \mathcal{T}(R)$ and $[\vec{u}^i], [\vec{v}^i] \in !\mathcal{T}(Q_i)$. We have now three subcases depending on the shape of R .

Subcase R variable, say $R = x$. Then $t' = s' = x$. *W.l.o.g.* $n \geq 1$, otherwise it is trivial that $t = s$. Now say $[\vec{u}^i] =: [u_1^i, \dots, u_{m_i}^i]$ and $[\vec{v}^i] =: [v_1^i, \dots, v_{m'_i}^i]$ for $i = 1, \dots, n$. By confluence we have $h \in \text{nf}_r(\vec{\lambda}\mu | x \text{nf}_r([\vec{u}^1]) \dots \text{nf}_r([\vec{u}^n]) |)$, so $h \in \text{nf}_r(\vec{\lambda}\mu | x [\vec{d}^1] \dots [\vec{d}^n] |)$ for some $d_j^i \in \text{nf}_r(u_j^i)$. Similarly, we get: $h \in \text{nf}_r(\vec{\lambda}\mu | x [\vec{d}'^1] \dots [\vec{d}'^n] |)$ for some $d_j^i \in \text{nf}_r(v_j^i)$. So it must be $m_i = m'_i$ ($i = 1, \dots, n$) and:

$$h = \vec{\lambda}\mu' | x [d_1^1, \dots, d_{m_1}^1] \dots [d_1^n, \dots, d_{m_n}^n] |$$

for some head $\vec{\lambda}\mu'$, some $d_j^i \in \text{nf}_r(u_j^i) \cap \text{nf}_r(v_{\sigma_i(j)}^i)$ and permutations σ_i on m_i elements. But $u_j^i, v_j^i \in \mathcal{T}(Q_i)$ and $\mathbf{ms}(u_j^i) < \mathbf{ms}(t)$ since u_j^i is a strict subterm of t . So we can apply the inductive hypothesis to each u_j^i and obtain $u_j^i = v_{\sigma_i(j)}^i$. Hence, $t = s$.

Subcase $R = (\lambda y.P)N$. It is the same argument as the following subcase, so we skip it.

Subcase $R = (\mu\gamma.\eta|P|)N$. Then $t' = (\mu\gamma.\eta|p|)[\vec{d}]$ and $s' = (\mu\gamma.\eta|p'|)[\vec{d}']$ for $p, p' \in \mathcal{T}(P)$ and $[\vec{d}], [\vec{d}'] \in !\mathcal{T}(N)$. By confluence on $\lambda\mu^r$ we have:

$$\text{nf}_r(t) = \text{nf}_r(\vec{\lambda}\mu | (\mu\gamma.\langle \eta | p \rangle_\gamma [\vec{d}]) [\vec{u}^1] \dots [\vec{u}^n] |).$$

So there is $h_1 \in \mu\gamma.\langle \eta | p \rangle_\gamma [\vec{d}]$ s.t. $h \in \text{nf}_r(\tilde{h}_1)$ where: $\tilde{h}_1 := \vec{\lambda}\mu | h_1 [\vec{u}^1] \dots [\vec{u}^n] |$. Analogously we find a $h_2 \in \mu\gamma.\langle \eta | p' \rangle_\gamma [\vec{d}']$ s.t. $h \in \text{nf}_r(\tilde{h}_2)$, where: $\tilde{h}_2 := \vec{\lambda}\mu | h_2 [\vec{v}^1] \dots [\vec{v}^n] |$. By Lemma 3.3 we have $h_1, h_2 \in \mathcal{T}(\mu\gamma.\langle \eta | P \rangle_\gamma N)$ and so \tilde{h}_1, \tilde{h}_2 belong to $\mathcal{T}(\vec{\lambda}\mu | (\mu\gamma.\langle \eta | P \rangle_\gamma N) Q_1 \dots Q_n |)$. This and the fact that $h \in \text{nf}_r(\tilde{h}_1) \cap \text{nf}_r(\tilde{h}_2)$ mean that \tilde{h}_1 satisfies both the hypotheses of the inductive hypothesis. Moreover, since $t' \rightarrow_{\mu^r} h_1 + \mathbb{T}$ for some sum \mathbb{T} , then $m(h_1) < m(t')$. And since the number of μ 's is constant under μ -reduction, $\text{deg}_\mu(t') = \text{deg}_\mu(h_1)$. Therefore we can apply Lemma 2.15(2) and obtain: $m(\tilde{h}_1) = m(\vec{\lambda}\mu | h_1 [\vec{u}^1] \dots [\vec{u}^n] |) < m(\vec{\lambda}\mu | t' [\vec{u}^1] \dots [\vec{u}^n] |) = m(t)$. So $\mathbf{ms}(\tilde{h}_1) < \mathbf{ms}(t)$ and we can safely apply the inductive hypothesis obtaining $\tilde{h}_1 = \tilde{h}_2$. Looking at the definition of

\tilde{h}_1, \tilde{h}_2 , we get $h_1 = h_2$ as well as $[\tilde{u}^i] = [\tilde{v}^i]$ ($i = 1, \dots, n$). But now we have:

$$\mu\gamma.\langle \eta | p \rangle_\gamma [\vec{d}] \ni h_1 = h_2 \in \mu\gamma.\langle \eta | p \rangle_\gamma [\vec{d}']$$

so Lemma 3.5 gives $p = p'$ and $[\vec{d}] = [\vec{d}']$, i.e. $t' = s'$. If we look at the shape of t, s , this last information together with $[\tilde{u}^1] = [\tilde{v}^1], \dots, [\tilde{u}^n] = [\tilde{v}^n]$, precisely means $t = s$. \square

We conclude with a useful property (Corollary 3.8). It follows from the following proposition, which in turn easily follows by Lemma 3.3.

Proposition 3.7. *If $\mathcal{T}(M) \ni t \rightarrow_{\text{base}} \mathbb{T}'$ then there is $N \in \lambda\mu$ s.t. $\mathbb{T}' \subseteq \mathcal{T}(N)$ and $M \rightarrow N$.*

Corollary 3.8. *For all $\mathbb{T} \subseteq \mathcal{T}(M)$, there exist $N \in \lambda\mu$ s.t. $M \rightarrow N$ and $\text{nf}_r(\mathbb{T}) \subseteq \mathcal{T}(N)$.*

Proof sketch. One first generalises Proposition 3.7 to sums (instead of a term t in the statement); then, we prove the desired result by induction on the length of a maximal reduction $\mathbb{T} \rightarrow_r \text{nf}_r(\mathbb{T})$. \square

3.2 The $\lambda\mu$ -theory $=_\tau$

Mimicking the definitions for λ -calculus we say that:

- Definition 3.9.**
1. An equivalence \mathcal{R} on $\lambda\mu$ is a congruence iff \mathcal{R} is contextual.
 2. A congruence \mathcal{R} is a $\lambda\mu$ -theory iff $\mathcal{R} \supseteq =_{\lambda\mu\rho}$.
 3. The term algebra of a $\lambda\mu$ -theory \mathcal{R} is the quotient $\lambda\mu/\mathcal{R}$. A $\lambda\mu$ -theory \mathcal{R} is non-trivial iff $\lambda\mu/\mathcal{R} \neq \{*\}$.

It is clear that $=_{\lambda\mu\rho}$ is a $\lambda\mu$ -theory. Now fix the equivalence $M =_\tau N$ iff $\text{NF}\mathcal{T}(M) = \text{NF}\mathcal{T}(N)$. Actually, $=_\tau$ is a non-trivial $\lambda\mu$ -theory. In fact, the contextuality follows immediately from the Theorem 3.2; the fact that it contains $=_{\lambda\mu\rho}$ easily follows from confluence and Proposition 3.4; and it is clearly non-trivial: $\lambda x.x \neq_\tau \emptyset =_\tau (\lambda x.xx)(\lambda x.xx) =: \Omega$.

In λ -calculus, $=_\tau$ is the λ -theory equating Böhm trees. In particular, it is sensible (i.e. it equates all unsolvables). We will see (Corollary 3.15) that in our case it is still sensible.

Definition 3.10. A $\lambda\mu$ -term M is a head normal form (hnf for short) iff there are no ρ -redexes in its head (remember Lemma 1.4) and it has a head variable. We define the exact same notion for $\lambda\mu^f$.

Definition 3.11. The head reduction is the partial function $H : \lambda\mu \rightarrow \lambda\mu$ obtained defining $H(M)$ via the following algorithm:

1. ρ -reduce the leftmost ρ -redex in the head of M , if any
2. otherwise, $\lambda\mu$ -reduce the head redex of M , if any
3. otherwise, $H(M)$ is not defined.

$H(M)$ is not defined iff M is a hnf. We say that head reduction starting on M terminates iff there is a (necessarily unique) $n \geq 0$ s.t. $H^n(M)$ is a hnf. Here we mean as usual that $H^0(M) := M$.

We extend the same definitions to resource terms, and set $H(t) := \emptyset$ whenever t is a hnf. Moreover, we set $H^0(t) := t \in 2\langle \Lambda^f \rangle$ and, for $n \geq 0$:

$$H^{n+1}(t) := \sum_{t_1 \in H(t)} \sum_{t_2 \in H(t_1)} \cdots \sum_{t_{n+1} \in H(t_n)} t_{n+1} \in 2\langle \Lambda^f \rangle.$$

We have $H^1(t) = H(t)$ and $H^{n+1}(t) = \sum_{t' \in H(t)} H^n(t')$.

Lemma 3.12. *If s only contains empty bags (if any) and $s \in \text{nf}_r(t)$, then $s \in H^n(t)$ for some $n \geq 0$.*

Proof sketch. If t is a hnf, $s \in \text{nf}_r(t)$ entails that t already contains only empty bags, as any eventual bag of s is empty and reductions cannot erase non-empty bags; but in a hnf the reduction can only take place inside some bag, so it must be $s = t$ and we are done. If t is not hnf, by confluence $t \rightarrow_r H(t) \rightarrow_r \text{nf}_r(t) \ni s$. So there is a $t_1 \in H(t)$ s.t. $s \in \text{nf}_r(t_1)$. Now we reason as in the beginning: if t_1 is hnf we are done; if t_1 is not, we repeat the argument finding some t_2 . By the well-foundedness of $\bar{m}(\cdot)$, we cannot repeat the argument forever and we must end on a hnf, so we conclude. \square

Set $H(\mathcal{T}(M)) := \bigcup_{t \in \mathcal{T}(M)} H(t) \subseteq \lambda\mu^f$. The following lemma is easy using Definition 3.10 and Lemma 3.3.

Lemma 3.13. *If $M \in \lambda\mu$ with $H(M)$ defined, we have:*

$$\mathcal{T}(H(M)) = H(\mathcal{T}(M)).$$

The following proposition shows that $\lambda\mu$ -calculus enjoys a notion of solvability analogue to the one of λ -calculus.

Proposition 3.14. *For $M \in \lambda\mu$, the following are equivalent:*

1. $M =_{\lambda\mu\rho} H$ with H hnf
2. Head reduction starting on M terminates
3. $\text{NF}\mathcal{T}(M) \neq \emptyset$.

We call $M \in \lambda\mu$ solvable iff it satisfies any of the previous equivalent conditions. Otherwise, M is called unsolvable.

Proof. (1 \Rightarrow 2). By confluence M and H have a common $\lambda\mu\rho$ -redex M_0 . Since H is a hnf, M_0 is too. Let s_0 be the unique resource $\lambda\mu$ -term in $\mathcal{T}(M_0)$ s.t. all its bags (if any) are empty. This term clearly exists. Note that, by construction, s_0 is r -normal. By repeatedly applying Proposition 3.4 one can check that we obtain an $s \in \mathcal{T}(M)$ s.t. $s_0 \in \text{nf}_r(s)$. Now, by Lemma 3.12, $s_0 \in H^n(s)$ for some $n \geq 0$. By repeatedly applying Lemma 3.13, we find that $s_0 \in H^n(\mathcal{T}(M)) = \mathcal{T}(H^n(M))$. Finally, since s_0 is a hnf, so it must be $H^n(M)$.

(2 \Rightarrow 3). Easy.

(3 \Rightarrow 1). If $\text{NF}\mathcal{T}(M) \neq \emptyset$ there is $t \in \mathcal{T}(M)$ s.t. $\text{nf}_r(t) \neq \emptyset$. By Corollary 3.8, $M \rightarrow N$ for some $N \in \lambda\mu$ s.t. $\text{nf}_r(t) \subseteq \mathcal{T}(N)$. So $\mathcal{T}(N)$ contains at least a hnf, and thus N must be a hnf too. \square

Corollary 3.15. *The $\lambda\mu$ -theory $=_\tau$ is sensible (that is, it equates all unsolvable terms).*

4 Applying the approximation theory

4.1 Stability

The Stability Property gives sufficient conditions for a context to commute with intersections in $\lambda\mu/\equiv_\tau$, i.e. (the intersections are defined below):

$$C(\bigcap_{i_1} N_{i_1}, \dots, \bigcap_{i_n} N_{i_n}) \equiv_\tau \bigcap_{i_1, \dots, i_n} C(N_{i_1}, \dots, N_{i_n}).$$

Given a non-empty subset $X \subseteq \lambda\mu$, call its \mathcal{T} -infimum the set $\bigcap X := \bigcap_{M \in X} \text{NF}\mathcal{T}(M) \subseteq \lambda\mu^f$. We say that X is *bounded* iff there exists an $L \in \lambda\mu$ such that $M \leq L$ for all $M \in X$. Write $M \equiv_\tau \bigcap X$ instead of $\text{NF}\mathcal{T}(M) = \bigcap X$. Observe that (in case it exists) an M s.t. $M \equiv_\tau \bigcap X$ need *not* to be unique, so $\bigcap X$ does not identify a unique λ -term.

Theorem 4.1 (Stability). *Let C be an n -ary $\lambda\mu$ -context and fix non empty bounded $X_1, \dots, X_n \subseteq \lambda\mu^f$. For all $M_1, \dots, M_n \in \lambda\mu$ s.t. $M_i \equiv_\tau \bigcap X_i$ ($i = 1, \dots, n$) we have:*

$$C(M_1, \dots, M_n) \equiv_\tau \bigcap_{\substack{N_1 \in X_1 \\ \dots \\ N_n \in X_n}} C(N_1, \dots, N_n).$$

Proof. Non-trivial, but exactly as done in [1] for λ -calculus (using Theorem 3.6). \square

Using the usual encoding of booleans and pairs (**True** := $\lambda xy.x$, **False** := $\lambda xy.y$, $\langle M, N \rangle := \lambda z.zMN$) we have the non-implementability of the following parallel-or.

Corollary 4.2. *There is no $\text{Por} \in \lambda\mu$ s.t. for all $M, N \in \lambda\mu$,*

$$\begin{cases} \text{Por}\langle M, N \rangle \equiv_\tau \text{True} & \text{if } M \not\equiv_\tau \Omega \text{ or } N \not\equiv_\tau \Omega \\ \text{Por}\langle M, N \rangle \equiv_\tau \Omega & \text{if } M \equiv_\tau N \equiv_\tau \Omega. \end{cases}$$

Proof. Otherwise, for the context $C := \text{Por } \xi$, by Theorem 4.1 we would have the contradiction: $\text{True} \equiv_\tau C(\langle \text{True}, \Omega \rangle) \cap C(\langle \Omega, \text{True} \rangle) \equiv_\tau C(\langle \text{True}, \Omega \rangle \cap \langle \Omega, \text{True} \rangle) \equiv_\tau C(\langle \Omega, \Omega \rangle) \equiv_\tau \Omega$. \square

4.2 The perpendicular Lines Property

The perpendicular lines Property (PLP for short) states that, fixed a term $\lambda z_1 \dots z_n.F \in \lambda\mu$, if the function $\vec{M} \in \lambda\mu^n/\equiv_\tau \rightarrow (\lambda \vec{z}.F)\vec{M} \in \lambda\mu/\equiv_\tau$ is constant on n “perpendicular lines” (in the sense of the statement, Theorem 4.4), then it is constant everywhere. Lemma 4.3 below is the crucial ingredient for the proof of PLP, and we use in it all the strong properties of resource approximation (linearity, SN and confluence).

Lemma 4.3. *Fix $\vec{z} := z_1, \dots, z_n$ distinct variables and let $t \in \lambda\mu^f$. Suppose that:*

- i. $\text{nf}_r(t) \neq 0$
- ii. *there is $F \in \lambda\mu$ s.t. $t \in \mathcal{T}(F)$*
- iii. *there are $\{M_{ij}\}_{1 \leq i \neq j \leq n} \subseteq \lambda\mu$ s.t. the function mapping $\vec{M} \in \lambda\mu^n/\equiv_\tau$ to $(\lambda \vec{z}.F)\vec{M} \in \lambda\mu/\equiv_\tau$ is constant on the*

following “perpendicular lines” of $\lambda\mu^n/\equiv_\tau$:

$$\begin{aligned} l_1 &= \{(Z, M_{12}, \dots, M_{1n}) \mid Z \in \lambda\mu\} \\ l_2 &= \{(M_{21}, Z, \dots, M_{2n}) \mid Z \in \lambda\mu\} \\ &\vdots \\ l_n &= \{(M_{n1}, \dots, M_{n(n-1)}, Z) \mid Z \in \lambda\mu\}. \end{aligned} \tag{1}$$

Then $\text{deg}_{z_1}(t) = \dots = \text{deg}_{z_n}(t) = 0$.

Proof. Induction on the size $\mathbf{ms}(t)$ of $t \in \lambda\mu^f$.

- Case $\mathbf{ms}(t) = (1, 0, 1)$. Then t is a variable (Corollary 2.16). If $t = z_i$ for some i then the i -th line of (1) gives the contradiction:

$$N_i \equiv_\tau (\lambda \vec{z}.z_i)M_{i1} \dots M_{i(i-1)}ZM_{i(i+1)} \dots M_{in} \equiv_\tau Z$$

for all $Z \in \lambda\mu$. Hence, it must be $\text{deg}_{z_1}(t) = \dots = \text{deg}_{z_n}(t) = 0$.

- Case $\mathbf{ms}(t) > (1, 0, 1)$. By (i) there is $u \in \text{nf}_r(t)$. As u is normal, it has shape: $u = \tilde{\lambda} \mu |y[\vec{u}^1] \dots [\vec{u}^m]|$ for some $m \geq 0$, some variable y , some normal bags $[\vec{u}^j]$, and where we have shorten, as before, a series $\lambda \vec{x}_1 \mu \alpha_{1.\beta_1} | \dots | \lambda \vec{x}_k \mu \alpha_{k.\beta_k} | \dots |$ of λ and μ abstraction by just $\tilde{\lambda} \mu | \dots |$. By (ii) $t \in \mathcal{T}(F)$, so that by Corollary 3.8 there is $Q \in \lambda\mu$ s.t. $F \twoheadrightarrow Q$ and $u \in \mathcal{T}(Q)$. So Q must have shape: $Q = \tilde{\lambda} \mu |yQ_1 \dots Q_m|$ for some Q_j 's in $\lambda\mu$ s.t. $[\vec{u}^j] \in !\mathcal{T}(Q_j)$ for all $j = 1, \dots, m$. Now there are two possibilities: either $y = z_i$ for some $i = 1, \dots, n$, either $y \neq z_i$ for all i .

Suppose $y = z_i$. Then, for $\vec{q} := q_1, \dots, q_m$ fresh variables, we can chose $Z := \lambda \vec{q}.\text{True} \in \lambda\mu$ (or $Z := \text{True}$ if $m = 0$) in the i -th line l_i of (1), and since by (iii) $\lambda \vec{z}.F$ is constant (mod \equiv_τ) on l_i , we can compute its value as:

$$\begin{aligned} &(\lambda \vec{z}.F)M_{i1} \dots M_{i(i-1)}(\lambda \vec{q}.\text{True})M_{i(i+1)} \dots M_{in} \\ &\equiv_\tau Q\{M_{i1}/z_1, \dots, (\lambda \vec{q}.\text{True})/z_i, \dots, M_{in}/z_n\} \\ &\equiv_\tau \tilde{\lambda} \mu |(\lambda \vec{q}.\text{True})\widetilde{Q}_{i1} \dots \widetilde{Q}_{im}| \\ &\equiv_\tau \tilde{\lambda} \mu |\text{True}| \end{aligned}$$

where we set $\widetilde{Q}_{ij} := Q_j\{M_{i1}/z_1, \dots, (\lambda \vec{q}.\text{True})/z_i, \dots, M_{in}/z_n\}$. The first equality holds because $F \twoheadrightarrow Q$ and \equiv_τ is finer than $=_{\lambda\mu\rho}$, and the second equality holds because $y = z_i$. In the same way, choosing $Z := \lambda \vec{q}.\text{False} \in \lambda\mu$ in l_i we find that the value (mod \equiv_τ) of $\lambda \vec{z}.F$ on l_i is $\tilde{\lambda} \mu |\text{False}|$. But this is impossible because $\text{True} \not\equiv_\tau \text{False}$.

Therefore, it must be $y \neq z_i$ for all i . Note that *w.l.o.g.* $m \geq 1$ (indeed if $m = 0$, from the fact that $y \neq z_i$ for all i we already get $\text{deg}_{z_i}(u) = 0$ and, as $u \in \text{nf}_r(t)$ and in $\lambda\mu^f$ one cannot erase non-empty bags, we are done). Now fix $i \in \{1, \dots, n\}$ and $Z', Z'' \in \lambda\mu$. Similarly as before, choosing $Z := Z'$ in l_i and using what we found so far, putting $Q'_{ij} := Q_j\{M_{i1}/z_1, \dots, Z'/z_i, \dots, M_{in}/z_n\}$, since $\lambda \vec{z}.F$ is constant (mod \equiv_τ) on l_i , we can compute its value as:

$$\begin{aligned} &(\lambda \vec{z}.F)M_{i1} \dots M_{i(i-1)}Z'M_{i(i+1)} \dots M_{in} \\ &\equiv_\tau Q\{M_{i1}/z_1, \dots, Z'/z_i, \dots, M_{in}/z_n\} \\ &\equiv_\tau \tilde{\lambda} \mu |yQ'_{i1} \dots Q'_{im}| \end{aligned}$$

where the last equality holds since y is *not* one of the z_i 's. Choosing Z'' instead of Z' and putting Q''_{ij} the same as Q'_{ij}

but with Z'' instead of Z' , one has that the value $(\text{mod } =_\tau)$ of $\lambda\vec{z}.F$ on l_i is $\vec{\lambda}\mu|yQ'_{i1} \dots Q'_{im}|$. So we have $\vec{\lambda}\mu|yQ'_{i1} \dots Q'_{im}| = \vec{\lambda}\mu|yQ''_{i1} \dots Q''_{im}|$, which easily entails: $Q'_{i1} =_\tau Q''_{i1}, \dots, Q'_{im} =_\tau Q''_{im}$. But by construction we have:

$$\begin{aligned} Q'_{ij} &=_\tau (\lambda\vec{z}.Q_j)M_{i1} \dots M_{i(i-1)}Z'M_{i(i+1)} \dots M_{in} \\ Q''_{ij} &=_\tau (\lambda\vec{z}.Q_j)M_{i1} \dots M_{i(i-1)}Z''M_{i(i+1)} \dots M_{in}. \end{aligned}$$

So if we remember that Z', Z'' were generic in $\lambda\mu$, the previous equalities $Q'_{ij} = Q''_{ij}$ precisely say that $\lambda\vec{z}.Q_j$ is constant on the line l_i . And since this holds for all $i = 1, \dots, n$, we have just found that $\lambda\vec{z}.Q_j$ satisfies (iii). And since we have equalities $Q'_{ij} = Q''_{ij}$ for all $j = 1, \dots, m$, we have that each $\lambda\vec{z}.Q_1, \dots, \lambda\vec{z}.Q_k$ satisfies (iii). We can now comfortably apply the induction hypothesis on any $s \in [\vec{u}^j]$. In fact, as $[\vec{u}^j]$ is normal, $\text{nf}_\tau(s) \neq 0$, i.e. (i); as $[\vec{u}^j] \in !\mathcal{T}(Q_j)$, we have $s \in \mathcal{T}(Q_j)$, i.e. (ii); and we just found that $\lambda\vec{z}.Q_j$ satisfies (iii); finally, s is a strict subterm of $u \in \text{nf}_\tau(t)$, thus (Corollary 2.16) $\text{ms}(s) < \text{ms}(u) \leq \text{ms}(t)$. Therefore, the inductive hypothesis gives $\text{deg}_{z_1}(s) = \dots = \text{deg}_{z_n}(s) = 0$. Since this is true for all s in all $[\vec{u}^j]$, $j = 1, \dots, m$, we get $\text{deg}_{z_1}(u) = \dots = \text{deg}_{z_n}(u) = 0$. And now $u \in \text{nf}_\tau(t)$ entails $\text{deg}_{z_1}(t) = \dots = \text{deg}_{z_n}(t) = 0$. \square

Theorem 4.4 (Perpendicular Lines Property). *Suppose that for some fixed $\{M_{ij}\}_{1 \leq i \neq j \leq n}$, $\{N_i\}_{1 \leq i \leq n} \subseteq \lambda\mu$, the system of equations:*

$$\begin{cases} (\lambda z_1 \dots z_n.F) Z M_{12} \dots M_{1n} =_\tau N_1 \\ (\lambda z_1 \dots z_n.F) M_{21} Z \dots M_{2n} =_\tau N_2 \\ \vdots \\ (\lambda z_1 \dots z_n.F) M_{n1} \dots M_{n(n-1)} Z =_\tau N_n \end{cases}$$

holds for all $Z \in \lambda\mu$. Then:

$$(\lambda z_1 \dots z_n.F)Z_1 \dots Z_n =_\tau N_1$$

for all $Z_1, \dots, Z_n \in \lambda\mu$.

Proof. It follows from Lemma 4.3 as done in [1]. \square

Corollary 4.5. *There is no $\text{Por}' \in \lambda\mu$ s.t. for all $Z \in \lambda\mu$ one has at the same time $\text{Por}' \text{ True } Z =_\tau \text{ True}$, $\text{Por}' Z \text{ True} =_\tau \text{ True}$, $\text{Por}' \text{ False } \text{ False} =_\tau \text{ False}$.*

The non existence of parallelism in $\lambda\mu$ -calculus is known as folklore via arguments involving stable models: here we proved it solely via Taylor expansion.

5 Conclusions and Future Works

In [23] Laurent studies the mathematics of (untyped) $\lambda\mu$ -calculus via its denotational semantics; this paper does it by developing a theory of program approximation based on Linear Logic resources. In particular, we proved that the approximation theory satisfies strong normalisation and confluence (non-trivial results in this setting), the Monotonicity Property, the Non-Interference Property, that it induces a sensible $\lambda\mu$ -theory, and that it can be used as a tool in order

to obtain the Stability Property and the Perpendicular Lines Property, and thus the impossibility of parallel computations in the language. A first natural question immediately arises:

1- Does Taylor expansion allow to find interesting properties *not* satisfied by λ -calculus, but that are enjoyed by the $\lambda\mu$ -calculus due to the presence of `callcc`?

For future investigations, we believe that it would be interesting to integrate this approach with the *differential* extension of $\lambda\mu$ -calculus defined in [30], via the mentioned translation $(\cdot)^{\flat}$, in order to explore quantitative properties.

The following two questions are maybe the most significant ones:

2- Does it makes sense to introduce Böhm trees for the $\lambda\mu$ -calculus? For instance, for the call-by-value λ -calculus, the Taylor expansion has provided in [17] invaluable guidance for finding a meaningful notion of trees satisfying Ehrhard and Regnier's *commutation formula*; the same methodology could maybe be applied here. However, in [7] it is shown that $\lambda\mu$ -calculus does *not* enjoy Böhm's separation property. David and Py's counterexample could hence be an indication that, instead, Böhm trees are not a "good" notion for $\lambda\mu$ -calculus. The best way of proceeding would be, in that case, to consider the natural extension of $\lambda\mu$ -calculus given by Saurin's $\Lambda\mu$ -calculus [28]. It was introduced precisely to satisfy Böhm's property and, as a matter of fact, in [28] Saurin proposes a definition of Böhm trees for his $\Lambda\mu$ -calculus.

3- Does $\Lambda\mu$ -calculus enjoys the same approximation theory developed in this paper? On one hand, many constructions we did in this chapter seem possible also in Saurin's calculus, on the other hand we used the fact that the number of μ 's in a term is the same as the named subterms, e.g. in Remark 2.7. In general, one could wonder which one, between $\lambda\mu$ and $\Lambda\mu$, should be the "canonical lambda-mu-calculus": from a proof-theoretical perspective $\lambda\mu$ -calculus precisely corresponds to Parigot's CD-derivations, but $\Lambda\mu$ -calculus satisfies more desirable properties (Böhm separation). Moreover, in [27], Saurin adapts usual techniques of λ -calculus to $\Lambda\mu$ -calculus: he studies the notion of solvability, proves a standardization theorem and studies more in detail the notion of Böhm trees. A very interesting future direction of research would be, hence, to develop our theory of resource approximation for Saurin's calculus, and study its relation with his theory of Böhm approximation. In any case, we look at the fact that the Taylor expansion works so nicely in $\lambda\mu$ -calculus – and this regardless of a notion of Böhm trees – as a *posteriori* confirmation of the high power of this form of approximation.

There are at least three other interesting points in relation with strictly related areas:

3- The $\lambda\mu$ -calculus can be translated in the λ -calculus via the CPS-translations (see e.g. de Groote's one in [8]). What does our theory of approximation correspond to under this translation?

4- In order to perform a deeper logical analysis, one should consider translations into Linear Logic. It is known from [22] that $\lambda\mu$ -calculus translates into polarized proof nets. Taylor expansion for proof-nets is possible, but the construction can be complex: in fact one of the interests in *directly* defining a Taylor expansion for a certain “ λ -calculus style” programming language (as we did for $\lambda\mu$ -calculus, and as one does for λ -calculus) is precisely to avoid that complexity. In our case we have just shown that, at the end of the day, the theory of resource approximation for $\lambda\mu$ -calculus can be developed with essentially the same methodology as in λ -calculus. This leads to asking what makes a Taylor expansion “easy”, and should be considered in relation to the already mentioned possibility of the existence of a coherence relation for which $\mathcal{T}(M)$ is a clique. This motivates an investigation of the complexity of the definition of a Taylor expansion of a programming language/proof system, which may be related to the notion of connectedness of proof-nets, whose study starts in [14]. Such question should be considered in relation with the so-called problem of the “inversion of Taylor expansion” [15, 16] and the problem of “injectivity” of denotational models (in particular, the relational one) for Linear Logic.

5- The $\lambda\mu$ -calculus is not the only way of extending the Curry-Howard correspondence to classical logic. Another notable one is the already mentioned *Krivine’s classical realizability*, which is a “machine to extract computational content from proofs + axioms” (for almost all mathematics, such as the one formalizable in ZF+AC, see [20]). There are translations between $\lambda\mu$ -calculus and Krivine’s calculus, and *vice-versa*. What does our work say about Krivine’s realizability?

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A TECHNICAL APPENDIX

Below, the technical appendix, where we report all the proofs not given in the main paper. The appendix is organised in five sections (A.1 to A.6). At the beginning of each of them, we indicate which are the results of the main paper that we are going to prove in it. Remark that at the very end (after the Appendix A.6) there are some figures to which we refer during the appendices.

A.1 APPENDIX - SECTION 2.1

We give proofs of Proposition 2.10, Proposition 2.11 and Corollary 2.13.

Proof of Proposition 2.10. If $t \rightarrow_{\lambda^r} t' + \mathbb{T}$ then $t = c(\langle \lambda x.s \rangle b_0)$ and $t' = c(\langle h \rangle)$ with $h = s\{u_{\sigma(1)}/x^{(1)}, \dots, u_{\sigma(n)}/x^{(n)}\} \in s\langle b_0/x \rangle$, for c a single-hole resource context, σ a permutation and $b_0 = [u_1, \dots, u_n]$. Call $k := \deg_{\mu}(t) = \deg_{\mu}(t')$, and let us use the same notation B_i^c as in the proof of Proposition 2.9. Then we have that $m(t')$ is the bag:

$$k - B_i^c * B_i^s * [d_{i'}(b) \mid b \text{ in some } u_i \text{ in } b_0]$$

and $m(t)$ is the bag:

$$m(t) = k - B_i^c * B_i^s * [d_i(b) \mid b \text{ in some } u_i \text{ in } b_0] * [d_i(b_0)].$$

Now, it is easily understood that if b occurs in c , or b occurs in s , then $d_{i'}(b) = d_i(b)$. Furthermore, for all b occurrence of bag in some $u_{\sigma(i)}$ belonging to b_0 , one has:

$$d_{i'}(b) = d_c(\xi) + d_s(x^{(i)}) + d_{u_{\sigma(i)}}(b) \geq d_c(\xi) + d_{u_{\sigma(i)}}(b) = d_i(b).$$

Thus: $m(t) \geq m(t') * [k - d_i(b_0)] > m(t')$. \square

Proof of Proposition 2.11. If $t \rightarrow_{\rho^r} t' + \mathbb{T}$ then $t = c(\langle h \rangle)$ and $t' = c(\langle h' \rangle)$, with $h = \mu_{\gamma.\alpha}[\mu\beta.\eta]s$ and $h' = \mu_{\gamma.\eta}|s|\{\alpha/\beta\}$ and c a single-hole resource context. Therefore:

$$m(t') = [\deg_{\mu}(t') - d_{i'}(b) \mid b \text{ in } c] * [\deg_{\mu}(t') - d_{i'}(b) \mid b \text{ in } s].$$

$$m(t) = [\deg_{\mu}(t) - d_i(b) \mid b \text{ in } c] * [\deg_{\mu}(t) - d_i(b) \mid b \text{ in } s].$$

First, remark that $\deg_{\mu}(t') = 1 + \deg_{\mu}(c) + \deg_{\mu}(s)$ and $\deg_{\mu}(t) = 2 + \deg_{\mu}(c) + \deg_{\mu}(s)$.

Also, notice as usual that if b occurs in c then $d_{i'}(b) = d_c(b) = d_i(b)$. Putting these things together we have that, if c contains at least one bag:

$$\begin{aligned} [\deg_{\mu}(t') - d_{i'}(b) \mid b \text{ in } c] &= (1 + \deg_{\mu}(s)) + m(c) \\ &< (2 + \deg_{\mu}(s)) + m(c) = [\deg_{\mu}(t) - d_i(b) \mid b \text{ in } c]. \end{aligned}$$

On the other hand, if c does not contain any bag, the previous multisets are both empty, thus equal.

Now let's see what happens if b occurs in s . We have: $d_{i'}(b) = 1 + d_c(\xi) + d_s(b)$ and $d_i(b) = 2 + d_c(\xi) + d_s(b)$. Therefore:

$$\begin{aligned} [\deg_{\mu}(t') - d_{i'}(b) \mid b \text{ in } s] &= (\deg_{\mu}(c) - d_c(\xi)) + m(s) \\ &= [\deg_{\mu}(t) - d_i(b) \mid b \text{ in } s]. \end{aligned}$$

These facts immediately imply that $m(t) \geq m(t')$. \square

Proof of Corollary 2.13. The only case in which $m(\cdot)$ may remain constant is along a ρ^r -reduction, but in this case $\deg_{\mu}(t)$ strictly decreases, so $t \rightarrow_r t' + \mathbb{T}$ entails $\tilde{m}(t) > \tilde{m}(t')$. Now one concludes as usual: if we associate the multiset $[\tilde{m}(t) \mid t \in \mathbb{T}]$ with any sum \mathbb{T} (in particular, a single element sum), it is immediate to see that if $\mathbb{T} \rightarrow_r \mathbb{S}$, then the multiset associated with \mathbb{T} is strictly smaller, in the multiset order, than the one associated with \mathbb{S} (the empty multiset 1 being associated with the empty sum 0). Therefore the well-foundedness of the multiset order (since the order on $\tilde{m}(\cdot)$ is well-founded) gives the non-existence of infinite (non-trivial) reductions. \square

A.2 APPENDIX - SECTION 2.2

We give complete proofs of Lemma 2.25 and Proposition 2.26.

In order to prove Lemma 2.25, we need a number of technical lemmas handling the interaction of two successive substitutions. The proofs of these lemmas are tedious and long inductions and we do not report them here. In the following, we will use Notation 2.24.

Lemma A.1. *Let $t \in \lambda\mu^r$ and $\alpha, \beta, \gamma, \eta$ names.*

If $\alpha \neq \eta, \beta \neq \eta$ and $\beta \neq \gamma$, then $t\{\alpha/\beta\}\{\gamma/\eta\} = t\{\gamma/\eta\}\{\alpha/\beta\}$.

The following lemma says how a renaming behaves with respect to the linear substitution and the linear named application.

Lemma A.2. *Let $t \in \lambda\mu^r$ and $[\vec{u}]$ a bag. Then:*

1. $t\langle[\vec{u}]/x\rangle^+ \{\alpha/\beta\} = t\{\alpha/\beta\}\langle[\vec{u}\{\alpha/\beta\}]/x\rangle^+$.
2. *If $\alpha \neq \gamma \neq \beta$ then:*
 $\langle t \rangle_{\gamma}^+ [\vec{u}] \{\alpha/\beta\} = \langle t \{\alpha/\beta\} \rangle_{\gamma}^+ [\vec{u}\{\alpha/\beta\}]$.
3. *If $\alpha \neq \gamma \neq \beta$ then:*
 $\langle \langle \eta | t \rangle_{\gamma}^+ [\vec{u}] \rangle \{\alpha/\beta\} = \langle \eta | t \{\alpha/\beta\} \rangle_{\gamma}^+ [\vec{u}\{\alpha/\beta\}]$.

The next lemma says how two linear substitutions operate when applied consecutively.

Lemma A.3. *Let $t \in \lambda\mu^r$, $[\vec{v}] = [v_1, \dots, v_n]$ and $[\vec{u}]$ bags and $y \neq x$ variables with y not occurring in $[\vec{u}]$. Then the sum $t\langle[\vec{v}]/y\rangle\langle[\vec{u}]/x\rangle$ is:*

$$\sum_W t\langle[\vec{w}^0]/x\rangle\langle[v_1\langle[\vec{w}^1]/x\rangle, \dots, v_n\langle[\vec{w}^n]/x\rangle]/y\rangle$$

with $W : (\vec{u}) \rightarrow \{0, \dots, n\}$.

The condition $x \neq y$ in the previous lemma cannot be suppressed: for instance for $x = y, t = x, [\vec{v}] = 1$ and $[\vec{u}] = [z]$, with $z \neq y$, the left hand-side of the equality becomes $x\langle 1/x \rangle^+ \langle [z]/x \rangle^+ = 0$ while the right hand-side becomes $x\langle [z]/x \rangle^+ \langle 1/x \rangle^+ = z$.

The next lemma says how a linear substitution operates on linear named application.

Lemma A.4. *Let $t \in \lambda\mu^r$, $[\vec{v}] = [v_1, \dots, v_n]$ and $[\vec{u}]$ bags and α a name with⁶ $\deg_{\alpha}([\vec{u}]) = 0$. Then the sum*

⁶By " $\deg_{\beta}([\vec{u}]) = 0$ " we mean that $\deg_{\beta}(u) = 0$ for all $u \in [\vec{u}]$.

$(\langle t \rangle_\alpha^+ [\vec{v}]) \langle [\vec{u}] / x \rangle$ is:

$$\sum_W \langle t \langle [\vec{w}^0] / x \rangle \rangle_\alpha^+ [v_1 \langle [\vec{w}^1] / x \rangle, \dots, v_n \langle [\vec{w}^n] / x \rangle]$$

with $W : (\vec{u}) \longrightarrow \{0, \dots, n\}$.

The following two remarks are easy:

Remark A.5. Let α be a name, $t \in \lambda\mu^r$ and $[\vec{u}], [\vec{v}]$ bags. If $\deg_\alpha([\vec{v}]) = 0$ then one has:

$$\langle t[\vec{v}] \rangle_\alpha^+ [\vec{u}] = (\langle t \rangle_\alpha^+ [\vec{u}]) [\vec{v}].$$

Remark A.6. Let $\alpha \neq \beta$ be names, $t \in \lambda\mu^r$ and $[\vec{v}] =: [v_1, \dots, v_n], [\vec{u}]$ bags. If $\deg_\beta(t) = 0$ then one has:

$$\langle \langle t \rangle_\alpha^+ [\vec{v}] \rangle_\beta^+ [\vec{u}] = \sum_W \langle t \rangle_\alpha^+ [v_1 \langle [\vec{w}^1] \rangle_\beta^+, \dots, v_n \langle [\vec{w}^n] \rangle_\beta^+].$$

with $W : (\vec{u}) \longrightarrow \{1, \dots, n\}$.

The next lemma says in which sense and when, in some cases, one can swap the order of two linear applications.

Lemma A.7. Let $t \in \lambda\mu^r$, $[\vec{u}], [\vec{v}]$ bags and $\alpha \neq \beta$ names.

1. If $\deg_\alpha([\vec{v}]) = 0$ and $\deg_\beta([\vec{u}]) = 0$, then:

$$\langle \langle t \rangle_\alpha^+ [\vec{u}] \rangle_\beta^+ [\vec{v}] = \langle \langle t \rangle_\beta^+ [\vec{v}] \rangle_\alpha^+ [\vec{u}].$$

2. If $\deg_\alpha([\vec{v}]) = 0$ then, taking δ a fresh name, one has:

$$\langle \langle t \rangle_\alpha^+ [\vec{u}] \rangle_\beta^+ [\vec{v}] = \sum_W \langle \langle t \rangle_\beta^+ [\vec{w}^1] \rangle_\alpha^+ [\vec{u} \{ \delta / \beta \}] \rangle_\delta^+ [\vec{w}^2] \{ \beta / \delta \}$$

with $W : (\vec{v}) \longrightarrow \{1, 2\}$.

3. If $\deg_\beta([\vec{u}]) = 0$ then, taking δ a fresh name, one has:

$$\langle \langle t \rangle_\alpha^+ [\vec{u}] \rangle_\beta^+ [\vec{v}] = \langle \langle t \rangle_\beta^+ [\vec{v} \{ \delta / \alpha \}] \rangle_\alpha^+ [\vec{u}] \{ \alpha / \delta \}.$$

The next lemma says how a linear named applications on a name operates on a renaming involving the same name.

Lemma A.8. Let $t \in \lambda\mu^r$, $[\vec{u}]$ a bag and $\alpha \neq \beta$ names with $\deg_\beta([\vec{u}]) = 0$. Then:

$$\langle t \{ \alpha / \beta \} \rangle_\alpha^+ [\vec{u}] = \sum_W \langle \langle t \rangle_\alpha^+ [\vec{w}^1] \rangle_\beta^+ [\vec{w}^2] \{ \alpha / \beta \}$$

with $W : (\vec{u}) \longrightarrow \{1, 2\}$.

The condition $\alpha \neq \beta$ in the previous lemma cannot be suppressed: for instance, for $t = \mu\gamma.\alpha|x|$ and $[\vec{u}] = 1$, the left hand-side of the equality becomes $\mu\gamma.\alpha|x|1$ while the right hand-side becomes:

$$\sum_{W:(\vec{u}) \rightarrow \{1,2\}} \mu\gamma.\alpha|x| [\vec{w}^1] [\vec{w}^2] = \mu\gamma.\alpha|x|11.$$

Also the condition $\deg_\beta([\vec{u}]) = 0$ cannot be suppressed: for instance, for $t = \mu\gamma.\alpha|x|$ and $[\vec{u}] = [\mu\gamma'.\beta|y|]$, the left hand-side becomes $\mu\gamma.\alpha|x|[\mu\gamma'.\beta|y|]$ while the right hand-side becomes $\mu\gamma.\alpha|x|[\mu\gamma'.\beta|y|1]$.

The following lemma says how a linear named application operates on a linear substitution.

Lemma A.9. Let $t \in \lambda\mu^r$, $[\vec{v}] =: [v_1, \dots, v_n], [\vec{u}]$ bags, x a variable and α a name s.t. $\deg_x([\vec{u}]) = 0$. Then $\langle t \langle [\vec{v}] / x \rangle \rangle_\alpha^+ [\vec{u}]$ is:

$$\sum_W (\langle t \rangle_\alpha^+ [\vec{w}^0]) \langle [v_1 \langle [\vec{w}^1] \rangle_\alpha^+, \dots, v_n \langle [\vec{w}^n] \rangle_\alpha^+] / x \rangle$$

with $W : (\vec{u}) \longrightarrow \{0, \dots, n\}$.

The condition $\deg_x([\vec{u}]) = 0$ in the previous lemma cannot be suppressed: for instance, for $t = \mu\gamma.\alpha|y|$ (with $y \neq x$), $[\vec{v}] = 1$ and $[\vec{u}] = [x]$, the left hand-side becomes $\mu\gamma.\alpha|y|x|$ while the right hand-side becomes 0.

The next lemma says how two linear named application operate consecutively.

Lemma A.10. Let $t \in \lambda\mu^r$, $\alpha \neq \gamma$ names and $[\vec{v}] =: [v_1, \dots, v_n], [\vec{u}]$ bags s.t. $\deg_\gamma([\vec{u}]) = 0$. Then:

$$\langle \langle t \rangle_\gamma^+ [\vec{v}] \rangle_\alpha^+ [\vec{u}] = \sum_W \langle \langle t \rangle_\alpha^+ [\vec{w}^0] \rangle_\gamma^+ [v_1 \langle [\vec{w}^1] \rangle_\alpha^+, \dots, v_n \langle [\vec{w}^n] \rangle_\alpha^+]$$

with $W : (\vec{u}) \longrightarrow \{0, \dots, n\}$.

Furthermore, the same holds also for named terms. That is, under the same hypothesis and for η a name, we have⁷ that the sum $\langle \langle \eta | t | \rangle_\gamma^+ [\vec{v}] \rangle_\alpha^+ [\vec{u}]$ is:

$$\sum_W \langle \langle \eta | t | \rangle_\alpha^+ [\vec{w}^0] \rangle_\gamma^+ [v_1 \langle [\vec{w}^1] \rangle_\alpha^+, \dots, v_n \langle [\vec{w}^n] \rangle_\alpha^+]$$

with $W : (\vec{u}) \longrightarrow \{0, \dots, n\}$.

The condition $\alpha \neq \gamma$ in the previous lemma cannot be suppressed: for instance, for $t = \mu\delta.\alpha|x|$, $[\vec{u}] = [\mu\delta.\delta|x|]$ and $[\vec{v}] = [\mu\delta.\delta|y|]$, the left hand-side becomes $\mu\delta.\alpha|x|[\mu\delta.\delta|y|][\mu\delta.\delta|x|]$ while the right hand-side becomes $\mu\delta.\alpha|x|[\mu\delta.\delta|x|][\mu\delta.\delta|y|]$.

Also the condition $\deg_\gamma([\vec{u}]) = 0$ in the previous lemma cannot be suppressed: for instance, for $t = \mu\delta.\alpha|x|$, $[\vec{u}] = [\mu\delta.\gamma|x|]$ and $[\vec{v}] = [\mu\delta.\delta|x|]$, the left hand-side becomes 0 while the right hand-side becomes $\mu\delta.\alpha|x|[\mu\delta.\gamma|x|[\mu\delta.\delta|x|]]$.

Remark A.11. Let $[p, \vec{q}]$ be a bag. Then, every function $W' : \{p, u_1, \dots, u_k\} \longrightarrow \{0, \dots, n\}$ is uniquely determined by the choice of $W'(p) \in \{0, \dots, n\}$ plus the choice of a function $W : \{u_1, \dots, u_k\} \longrightarrow \{0, \dots, n\}$. This is reflected in the equality $(n+1)^{k+1} = (n+1)^k \cdot (n+1)$. Now, for $0 \leq i, j \leq n$, let us set $[p]_i^j$ the singleton multiset $[p]$ if $i = j$, and the empty multiset 1 if $i \neq j$. Therefore, the w.c. of $[p, \vec{q}]$ generated by a $W' : \{p, u_1, \dots, u_k\} \longrightarrow \{0, \dots, n\}$ is of shape $([\vec{w}^0] * [p]_0^{W'(p)}, \dots, [\vec{w}^n] * [p]_n^{W'(p)})$, for $([\vec{w}^0], \dots, [\vec{w}^n])$ a w.c. of $[\vec{q}]$ generated by a $W : \{u_1, \dots, u_k\} \longrightarrow \{0, \dots, n\}$.

In the main paper, there is a sketch of the proof of Lemma 2.25 with the most important cases. As declared at the beginning of this Appendix section, we give below its complete proof.

⁷The following is just an equality between sets of words – the named terms.

Proof of Lemma 2.25. (1). Induction on s :

Case s variable: impossible.

Case $s = \lambda y.s'$: straightforward by inductive hypothesis.

Case $s = \mu\gamma.\eta|s'|$: we have two subcases⁸:

Subcase $s \rightarrow_r^+ \mathbb{S}$ is performed by reducing s' :

then $\mathbb{S} = \mu\gamma.\eta|S'|$ with $s' \rightarrow_r^+ S'$, and so (regardless of whether η equals γ or not):

$$\begin{aligned} s\{\alpha/\beta\} &= \mu\gamma.\delta_{\eta}^{\alpha}(\beta)|s'\{\alpha/\beta\}| \rightarrow_r^+ \mu\gamma.\delta_{\eta}^{\alpha}(\beta)|S'\{\alpha/\beta\}| \\ &= \mu\gamma.\eta|S'\{\alpha/\beta\} = \mathbb{S}\{\alpha/\beta\} \end{aligned}$$

where we used the inductive hypothesis.

Subcase $s' = \mu\gamma'.\eta'|s''|$ and $s \rightarrow_r^+ \mathbb{S}$ is performed by reducing its leftmost ρ -redex:

Then $s = \mu\gamma.\eta|\mu\gamma'.\eta'|s''|$, $\mathbb{S} = \mu\gamma.\eta'|s''|\{\eta/\gamma'\}$ and we have four sub-subcases:

Sub-subcase $\eta = \beta$ and $\eta' = \beta$: then

$$\mathbb{S}\{\alpha/\beta\} = \mu\gamma.\beta|s''|\{\alpha/\beta, \alpha/\gamma'\} = \mu\gamma.\alpha|s''|\{\alpha/\beta, \alpha/\gamma'\}|$$

and:

$$\begin{aligned} s\{\alpha/\beta\} &= \mu\gamma.\alpha|\mu\gamma'.\eta'|s''|\{\alpha/\beta\}| \\ &\rightarrow_{\rho}^+ \mu\gamma.\alpha|s''|\{\alpha/\beta\}|\{\alpha/\gamma'\} \\ &= \mu\gamma.\alpha|s''|\{\alpha/\beta, \alpha/\gamma'\}| = \mathbb{S}\{\alpha/\beta\}. \end{aligned}$$

Sub-subcase $\eta = \beta$ and $\eta' \neq \beta$: then

$$\mathbb{S}\{\alpha/\beta\} = \mu\gamma.\eta'|s''|\{\alpha/\beta, \alpha/\gamma'\} = \mu\gamma.\delta_{\eta'}^{\alpha}(\gamma')|s''|\{\alpha/\beta, \alpha/\gamma'\}|$$

and:

$$\begin{aligned} s\{\alpha/\beta\} &= \mu\gamma.\alpha|\mu\gamma'.\eta'|s''|\{\alpha/\beta\}| \\ &\rightarrow_{\rho}^+ \mu\gamma.\eta'|s''|\{\alpha/\beta\}|\{\alpha/\gamma'\} \\ &= \mu\gamma.\delta_{\eta'}^{\alpha}(\gamma')|s''|\{\alpha/\beta, \alpha/\gamma'\}| = \mathbb{S}\{\alpha/\beta\}. \end{aligned}$$

Sub-subcase $\eta \neq \beta$ and $\eta' = \beta$: then

$$\mathbb{S}\{\alpha/\beta\} = \mu\gamma.\beta|s''|\{\eta/\gamma'\}|\{\alpha/\beta\} = \mu\gamma.\alpha|s''|\{\eta/\gamma'\}|\{\alpha/\beta\}|$$

and:

$$\begin{aligned} s\{\alpha/\beta\} &= \mu\gamma.\eta|\mu\gamma'.\eta'|s''|\{\alpha/\beta\}| \\ &\rightarrow_{\rho}^+ \mu\gamma.\alpha|s''|\{\alpha/\beta\}|\{\eta/\gamma'\} \\ &= \mu\gamma.\alpha|s''|\{\alpha/\beta\}|\{\eta/\gamma'\} \end{aligned}$$

and the result follows by Lemma A.1.

Sub-subcase $\eta \neq \beta$ and $\eta' \neq \beta$: then $\mathbb{S}\{\alpha/\beta\}$ is the term:

$$\mu\gamma.\delta_{\eta'}^{\alpha}(\gamma')|s''|\{\eta/\gamma'\}|\{\alpha/\beta\} = \mu\gamma.\delta_{\eta'}^{\alpha}(\gamma')|s''|\{\eta/\gamma'\}|\{\alpha/\beta\}|$$

and:

$$\begin{aligned} s\{\alpha/\beta\} &= \mu\gamma.\eta|\mu\gamma'.\eta'|s''|\{\alpha/\beta\}| \\ &\rightarrow_{\rho}^+ \mu\gamma.\eta'|s''|\{\alpha/\beta\}|\{\eta/\gamma'\} \\ &= \mu\gamma.\delta_{\eta'}^{\alpha}(\gamma')|s''|\{\alpha/\beta\}|\{\eta/\gamma'\} \end{aligned}$$

and the result follows by Lemma A.1.

Case $s = s'[\vec{v}]$: we have four subcases:

Subcase $s \rightarrow_r^+ \mathbb{S}$ is performed by reducing the s' : straightforward by inductive hypothesis.

Subcase $s \rightarrow_r^+ \mathbb{S}$ is performed by reducing one $v_i \in [\vec{v}]$: straightforward by inductive hypothesis.

⁸In the following we do not explicitly say it, but of course we will take bound names different from α and β , as well as from other bound names.

Subcase $s' = \lambda x.s''$ and $s \rightarrow_r^+ \mathbb{S}$ is performed by reducing the λ -redex s : then $\mathbb{S} = s''\langle[\vec{v}]/x\rangle^+$ and:

$$\begin{aligned} s\{\alpha/\beta\} &= (\lambda x.s''\{\alpha/\beta\})[\vec{v}\{\alpha/\beta\}] \\ &\rightarrow_{\lambda^r} s''\{\alpha/\beta\}\langle[\vec{v}\{\alpha/\beta\}]/x\rangle^+ \\ &= s''\langle[\vec{v}]/x\rangle^+\{\alpha/\beta\} = \mathbb{S}\{\alpha/\beta\} \end{aligned}$$

where the second-last equality holds thanks to Lemma A.2.

Subcase $s' = \mu\gamma.\eta|s''|$ and $s \rightarrow_r^+ \mathbb{S}$ is performed by reducing the μ -redex s : then $\mathbb{S} = \mu\gamma.\langle\eta|s''|\rangle_{\gamma}^+[\vec{v}]$ and:

$$\begin{aligned} s\{\alpha/\beta\} &= (\mu\gamma.\delta_{\eta}^{\alpha}(\beta)|s''|\{\alpha/\beta\}|)[\vec{v}\{\alpha/\beta\}] \\ &\rightarrow_{\mu^r} \mu\gamma.\langle\delta_{\eta}^{\alpha}(\beta)|s''|\{\alpha/\beta\}|\rangle_{\gamma}^+[\vec{v}\{\alpha/\beta\}] \\ &= \mu\gamma.\langle\eta|s''|\rangle_{\gamma}^+\{\alpha/\beta\}[\vec{v}\{\alpha/\beta\}] \end{aligned}$$

and the equality with $\mathbb{S}\{\alpha/\beta\}$ follows from Lemma A.2.

(2). Induction on t . The only non-trivial case is⁹ $t = v_0[v_1, \dots, v_n]$. In this case, by Remark A.11, we have that $(v_0[v_1, \dots, v_n])\langle[s, \vec{u}]/x\rangle^+$ is:

$$\sum_W \sum_{j=0}^n (v_0\langle[\vec{w}^0] * [s]_0^j/x\rangle) [\dots, v_i\langle[\vec{w}^i] * [s]_i^j/x\rangle, \dots].$$

where $W : (\vec{u}) \rightarrow \{1, \dots, n\}$. Fix now a $W : (\vec{u}) \rightarrow \{1, \dots, n\}$ (together with its generated w.c.) and consider each of the $n+1$ elements of the sum on j . We write the case for $j=0$, but the other cases are exactly the same. Since $j=0$, the element is $(v_0\langle[\vec{w}^0] * [s]/x\rangle) [\dots, v_i\langle[\vec{w}^i]/x\rangle, \dots]$ and by inductive hypothesis we have:

$$\begin{aligned} &(v_0\langle[\vec{w}^0] * [s]/x\rangle) [\dots, v_i\langle[\vec{w}^i] * [s]/x\rangle, \dots] \\ &\rightarrow_r^+ (v_0\langle[\vec{w}^0] * [\mathbb{S}]/x\rangle) [\dots, v_i\langle[\vec{w}^i]/x\rangle, \dots]. \end{aligned}$$

Now summing up all the elements for $j=0, \dots, n$ and $W : (\vec{u}) \rightarrow \{1, \dots, n\}$ we obtain the following sum:

$$\sum_W \sum_{j=0}^n (v_0\langle[\vec{w}^0] * [\mathbb{S}]_0^j/x\rangle) [\dots, v_i\langle[\vec{w}^i] * [\mathbb{S}]_i^j/x\rangle, \dots].$$

Using again Remark A.11, the above sum becomes:

$$(v_0[v_1, \dots, v_n])\langle[\mathbb{S}, \vec{u}]/x\rangle^+$$

which is the desired result.

(3). Induction on s .

Case s variable: impossible.

Case $s = \lambda$ -abstraction: straightforward by inductive hypothesis.

Case $s = \mu\alpha.\beta|s'|$: we have two subcases:

Subcase $s \rightarrow_r^+ \mathbb{S}$ is performed by reducing s' : immediate.

Subcase $s' = \mu\gamma.\eta|s''|$ and $s \rightarrow_r^+ \mathbb{S}$ is performed by reducing its leftmost ρ -redex:

⁹In the case $t = y \neq x$, and if $\mathbb{S} = 0$ and \vec{u} is empty, note that $t\langle[\mathbb{S}, \vec{u}]/x\rangle^+ = y\langle[0]/x\rangle^+ = \sum_{s'=0} y\langle[s']/x\rangle^+ = 0$ (and *not* $y\langle[0]/x\rangle^+ = y(1/x)^+ = y^n$), so the result still holds in this case.

then $\mathbb{S} = \mu\alpha.\eta|s''|\{\beta/\gamma\}$ and (call $[\vec{u}] = [u_1, \dots, u_k]$)

$$\begin{aligned} s\langle[\vec{u}]/x\rangle^+ &= \mu\alpha.\beta|\mu\gamma.\eta|s''\langle[\vec{u}]/x\rangle^+| \\ &\rightarrow_{\rho^t} \mu\alpha.\eta|s''\langle[\vec{u}]/x\rangle^+|\{\beta/\gamma\} \\ &= \mu\alpha.\eta|s''|\langle[\vec{u}]/x\rangle^+|\{\beta/\gamma\} \\ &= \mu\alpha.\eta|s''|\{\beta/\gamma\}\langle[\vec{u}]/x\rangle^+ \\ &= \mathbb{S}\langle[\vec{u}]/x\rangle^+. \end{aligned}$$

Case $s = s'[\vec{v}]$: we have four subcases:

Subcase $s \rightarrow_{\tau}^+ \mathbb{S}$ is performed by reducing s' : straightforward by inductive hypothesis.

Subcase $s \rightarrow_{\tau}^+ \mathbb{S}$ is performed by reducing the v_i in $[\vec{v}]$: straightforward by inductive hypothesis.

Subcase $s' = \lambda y.s''$ and $s \rightarrow_{\tau}^+ \mathbb{S}$ is performed by reducing the λ -redex s : follows by Lemma A.3 (where $x \neq y$ because y is bound and x is fixed).

Subcase $s' = \mu\alpha.\beta|s''|$ and $s \rightarrow_{\tau}^+ \mathbb{S}$ is performed by reducing the μ -redex s : follows by Lemma A.4 (where $\deg_{\alpha}([\vec{u}]) = 0$ because α is bound and $[\vec{u}]$ is fixed).

(4). Induction on t . The non-trivial cases are $t = v_0[v_1, \dots, v_n]$ and $t = \mu\gamma.\eta|t'|$. Both are done following the same argument as we did in point (2); apply Remark A.11, then the inductive hypothesis and thus close the argument by applying Remark A.11.

(5). It is immediate discriminating the cases $\alpha = \beta$ and $\alpha \neq \beta$, and concluding by point (4).

(6). Induction on $s \in \lambda\mu$.

Case $s = \text{variable}$. Impossible.

Case $s = \lambda$ -abstraction. Straightforward by inductive hypothesis.

Case $s = \mu\beta.\gamma|s'|$: we have two subcases:

Subcase $s \rightarrow_{\tau}^+ \mathbb{S}$ is performed by reducing s' : then $\mathbb{S} = \mu\beta.\gamma|\mathbb{S}'|$ with $s' \rightarrow_{\tau} \mathbb{S}'$. We have $\langle s \rangle_{\alpha}^+[\vec{u}] = \mu\beta.\langle \gamma|s'| \rangle_{\alpha}^+[\vec{u}]$. Remark that we *cannot* immediately apply the inductive hypothesis on $\gamma|s'|$, simply because $\gamma|s'| \notin \lambda\mu^t$ (it is a named term). However we can split in the two subcases whether $\gamma = \alpha$ or $\gamma \neq \alpha$ and now in both subcases we can conclude straightforwardly by inductive hypothesis.

Subcase $s' = \mu\gamma'.\eta|s''|$ (with $\gamma \neq \gamma'$) and $s \rightarrow_{\tau}^+ \mathbb{S}$ is performed by reducing its leftmost ρ -redex: then $\mathbb{S} = \mu\beta.\eta|s''|\{\gamma/\gamma'\}$. We split in two sub-subcases:

Sub-subcase $\alpha \neq \gamma$:

$$\begin{aligned} \langle s \rangle_{\alpha}^+[\vec{u}] &= \mu\beta.\gamma|\mu\gamma'.\langle \eta|s''| \rangle_{\alpha}^+[\vec{u}]| \\ &\rightarrow_{\rho^t} \mu\beta.\langle \eta|s''| \rangle_{\alpha}^+[\vec{u}]|\{\gamma/\gamma'\} \\ &= \langle \mu\beta.\eta|s''|\{\gamma/\gamma'\} \rangle_{\alpha}^+[\vec{u}] \quad (\text{by Lemma A.2} \\ &\quad \text{plus } \deg_{\gamma'}([\vec{u}]) = 0) \\ &= \langle \mathbb{S} \rangle_{\alpha}^+[\vec{u}]. \end{aligned}$$

Sub-subcase $\alpha = \gamma$. We have:

$$\begin{aligned} \langle s \rangle_{\alpha}^+[\vec{u}] &= \sum_W \mu\beta.\alpha|(\mu\gamma'.\langle \eta|s''| \rangle_{\alpha}^+[\vec{w}^1])[\vec{w}^2]| \\ &\quad \text{where } W : (\vec{u}) \rightarrow \{1, 2\} \\ &\rightarrow_{\mu^t} \sum_W \mu\beta.\alpha|\mu\gamma'.\langle \langle \eta|s''| \rangle_{\alpha}^+[\vec{w}^1] \rangle_{\gamma'}^+[\vec{w}^2]| \\ &\rightarrow_{\rho^t} \sum_W \mu\beta.\langle \langle \eta|s''| \rangle_{\alpha}^+[\vec{w}^1] \rangle_{\gamma'}^+[\vec{w}^2]|\{\alpha/\gamma'\} \\ &= \sum_W \langle \langle \mu\beta.\eta|s''| \rangle_{\alpha}^+[\vec{w}^1] \rangle_{\gamma'}^+[\vec{w}^2]|\{\alpha/\gamma'\} \\ &= \langle \mu\beta.\eta|s''|\{\alpha/\gamma'\} \rangle_{\alpha}^+[\vec{u}] \quad (\text{by Lemma A.8}) \\ &= \langle \mathbb{S} \rangle_{\alpha}^+[\vec{u}] \quad (\text{since } \alpha = \gamma). \end{aligned}$$

Case $s = s'[v_1, \dots, v_n]$: we have four subcases:

Subcase $s \rightarrow_{\tau}^+ \mathbb{S}$ is performed by reducing s' : straightforward by inductive hypothesis.

Subcase $s \rightarrow_{\tau}^+ \mathbb{S}$ is performed by reducing the v_i in $[\vec{v}]$: straightforward by inductive hypothesis.

Subcase $s' = \lambda x.s''$ and $s \rightarrow_{\tau}^+ \mathbb{S}$ is performed by reducing the λ -redex s . Then $\mathbb{S} = s''\langle[\vec{v}]/x\rangle$ and we have:

$$\begin{aligned} \langle s \rangle_{\alpha}^+[\vec{u}] &= \sum_W (\lambda x.\langle s'' \rangle_{\alpha}^+[\vec{w}^0])[\dots, \langle v_i \rangle_{\alpha}^+[\vec{w}^i], \dots] \\ &\quad \text{with } W : (\vec{u}) \rightarrow \{0, \dots, n\} \\ &\rightarrow_{\lambda^t} \sum_W (\langle s'' \rangle_{\alpha}^+[\vec{w}^0])[\dots, \langle v_i \rangle_{\alpha}^+[\vec{w}^i], \dots]/x \\ &= \langle s''\langle[\vec{v}]/x\rangle \rangle_{\alpha}^+[\vec{u}] \quad (\text{by Lemma A.9}) \\ &= \langle \mathbb{S} \rangle_{\alpha}^+[\vec{u}]. \end{aligned}$$

Subcase $s' = \mu\gamma.\eta|s''|$ (with $\gamma \neq \alpha$) and $s \rightarrow_{\tau}^+ \mathbb{S}$ is performed by reducing the μ -redex s . Then $\mathbb{S} = \mu\gamma.\langle \eta|s''| \rangle_{\gamma}^+[\vec{v}]$ and we have:

$$\begin{aligned} \langle s \rangle_{\alpha}^+[\vec{u}] &= \sum_W (\langle \mu\gamma.\eta|s''| \rangle_{\alpha}^+[\vec{w}^0])[\dots, \langle v_i \rangle_{\alpha}^+[\vec{w}^i], \dots] \\ &\quad \text{with } W : (\vec{u}) \rightarrow \{0, \dots, n\} \\ &\rightarrow_{\mu^t} \mu\gamma.\sum_W (\langle \eta|s''| \rangle_{\alpha}^+[\vec{w}^0])[\dots, \langle v_i \rangle_{\alpha}^+[\vec{w}^i], \dots] \\ &= \mu\gamma.\langle \langle \eta|s''| \rangle_{\gamma}^+[\vec{v}] \rangle_{\alpha}^+[\vec{u}] \quad (\text{by Lemma A.10}) \\ &= \langle \mathbb{S} \rangle_{\alpha}^+[\vec{u}]. \end{aligned}$$

(7). It is immediate by discriminating the cases $\alpha = \beta$ and $\alpha \neq \beta$, and then concluding by point (6). \square

In the main paper, there is a sketch of the proof of Proposition 2.26 with the most important cases. As declared at the beginning of this Appendix section, we give below its complete proof.

Proof of Proposition 2.26. We show, by induction on a single-hole resource context c , that if:

$$t \rightarrow_{\text{base}^t}^+ \mathbb{T} \text{ and } c(t) \rightarrow_{\tau}^+ \mathbb{T}_2$$

then there is $\mathbb{T}' \in \mathbb{N}\langle \lambda\mu^t \rangle$ s.t.

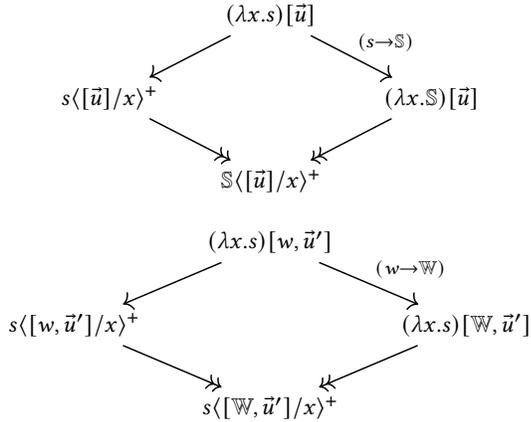
$$c(\mathbb{T}) \rightarrow_{\tau}^+ \mathbb{T}' \stackrel{+}{\leftarrow} \mathbb{T}_2.$$

In all the following diagrams we write “ \rightarrow ” but of course we mean “ \rightarrow_r^+ ”.

(1). Case $c = \xi$.

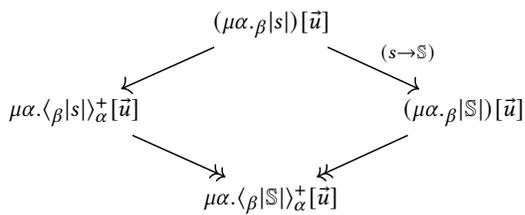
So $c(t) = t \rightarrow_{\text{base}^r}^+ \mathbb{T}$ and we only have the three base-cases of Definition 2.17. In all the diagrams of this case, when not explicitly said differently or not clear by an easy reduction, the bottom-left reduction follows from Lemma 2.25 and the bottom-right from Remark 2.23.

Subcase $t = (\lambda x.s)[\bar{u}]$ and $\mathbb{T} = s\langle[\bar{u}]/x\rangle^+$. Then $c(t) = t \rightarrow_r^+ \mathbb{T}_2$ (on a different redex than t) can only be performed either by reducing s , or by reducing an element w of $[\bar{u}]$. We have thus the following two diagrams:

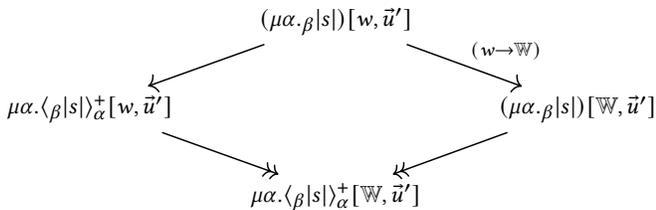


Subcase $t = (\mu\alpha.\beta|s|)[\bar{u}]$ and $\mathbb{T} = \mu\alpha.\langle\beta|s|\rangle_\alpha^+[\bar{u}]$. Then $c(t) = t \rightarrow_r^+ \mathbb{T}_2$ (on a different redex than t) can only be performed either by reducing s , or by reducing an element w of $[\bar{u}]$, or if $s = \mu\gamma.\eta|s'|$ and we reduce the ρ -redex $\dots_\beta|\mu\gamma.\dots|$. In the latter case we split into the case $\alpha \neq \beta$, the case $\alpha = \beta, \gamma \neq \eta, \eta = \alpha$, the case $\alpha = \beta, \gamma \neq \eta, \eta \neq \alpha$, and the case $\alpha = \beta, \eta = \gamma$ (with necessary $\gamma \neq \alpha$).

The first case is given by the diagram:



The second case is given by the diagram:



The third case is given by the diagram in Figure 3.

The fourth case is given by the diagram in Figure 4. The bottom right equality holds because, by Lemma A.8, Remark A.5

and since $\text{deg}_\gamma([\bar{u}]) = 0$, each addend of the bottom right sum is:

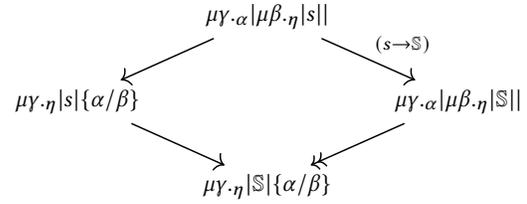
$$\begin{aligned} & \mu\alpha.\alpha|\langle\langle s' \{ \alpha / \gamma \} \rangle_\alpha^+ [\bar{w}^0]\rangle[\bar{w}^1]| \\ &= \sum_{D: (\bar{w}^0) \rightarrow \{1,2\}} \mu\alpha.\alpha|\langle\langle s' \rangle_\alpha^+ [\bar{d}^1]\rangle_\gamma^+ [\bar{d}^0]\rangle[\bar{w}^1]|\{ \alpha / \gamma \} \\ &= \sum_D \mu\alpha.\alpha|\langle\langle s' \rangle_\alpha^+ [\bar{d}^1]\rangle[\bar{w}^1]\rangle_\gamma^+ [\bar{d}^0]|\{ \alpha / \gamma \} \end{aligned}$$

and since we are then summing up on all possible $W : (\bar{u}) \rightarrow \{1,2\}$, the resulting sum is the same as the one at the bottom of the above diagram.

The fifth case is given by the diagram in Figure 5. The bottom right equality holds by Lemma A.8 and because $\text{deg}_\gamma([\bar{u}]) = 0$.

The sixth case is given by the diagram in Figure 6. The bottom right equality holds by Lemma A.8 and because $\text{deg}_\gamma([\bar{u}]) = 0$.

Subcase $t = \mu\gamma.\alpha|\mu\beta.\eta|s||$ and $\mathbb{T} = \mu\gamma.\eta|s|\{ \alpha / \beta \}$. Then $c(t) = t \rightarrow_r^+ \mathbb{T}_2$ (on a different redex than t) can be only performed either by reducing s , or if $s = \mu\gamma'.\eta'|s'|$ and we reduce the ρ -redex $\dots_\eta|\mu\gamma'.\dots|$. In the first case the diagram is:



In the second case the diagram is given in Figure 7. In order to prove that the second diagram holds, let us prove the equality in the last diagram: we first show that $s' \{ \alpha / \beta \} \{ \delta_0 / \gamma' \} = s' \{ \eta / \gamma' \} \{ \alpha / \beta \}$ and then that $\delta_1 = \delta_2$. For the former equality, we have:

if $\beta \neq \eta$, then $\delta_0 = \eta$ and the equality follows from Lemma A.1;

if $\beta = \eta$ then $\delta_0 = \alpha$ and the equality holds because both renamings $\{ \alpha / \beta \} \{ \delta_0 / \gamma' \}$ and $\{ \eta / \gamma' \} \{ \alpha / \beta \}$ coincide with the unique renaming $\{ \alpha / (\beta, \gamma') \}^{10}$.

For the latter equality, we have:

if $\gamma' = \eta'$: then $\delta'_2 = \eta$ and thus $\delta_2 = \delta_0$. Remark that it must be $\beta \neq \eta'$, because otherwise $\beta = \gamma'$ which is impossible. This means that $\delta'_1 = \eta'$. But then $\delta_1 = \delta_0$, so we are done.

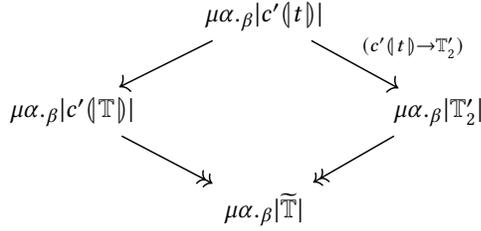
if $\gamma' \neq \eta'$: then $\delta'_2 = \eta'$ and thus $\delta_2 = \delta'_1$. Now if $\beta \neq \eta'$ then $\delta'_1 = \eta'$ and thus $\delta_1 = \eta'$; if $\beta = \eta'$ then $\delta'_1 = \alpha$ and thus $\delta_1 = \delta_\alpha^{\delta_0}(\gamma') = \alpha$, because it cannot be $\gamma' = \alpha$. If we read what we just found about δ_1 , it precisely says that $\delta_1 = \delta'_1$ so we are done.

(2). Case $c = \mu\alpha.\beta|c'|$ with either $c'(\dagger)$ not a μ -abstraction, or $c = \lambda x.c'$.

Then the reduction $c(\dagger) \rightarrow \mathbb{T}_2$ can only be performed via a reduction $c'(\dagger) \rightarrow \mathbb{T}'_2$. Both the diagrams for the two cases of c have the exact same shape; let us only give the one for

¹⁰We mean here that both β and γ' get renamed with α .

$c = \mu\alpha.\beta|c'|$:



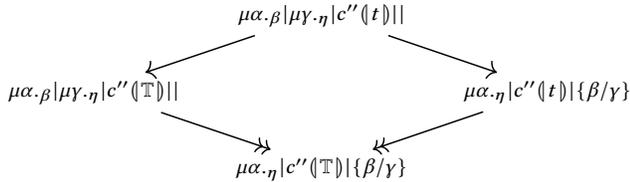
where a sum $\widetilde{\mathbb{T}}$ s.t. $c'(|\mathbb{T}|) \rightarrow \widetilde{\mathbb{T}} \leftarrow \mathbb{T}'_2$ is given by inductive hypothesis on c' .

(3). Case $c = \mu\alpha.\beta|c'|$, with $c'(|t|)$ a μ -abstraction.

Then either $c' = \xi$ and $t = \mu\gamma.\eta|t'_0|$, or $c' = \mu\gamma.\eta|c''|$.

In the first case, since by hypothesis $t \rightarrow_{\text{base}^r}^+ \mathbb{T}$, it must be $t = \mu\gamma.\eta|\mu\gamma'.\eta'|t'|$ and $\mathbb{T} = \mu\gamma.\eta'|t'|\{\eta/\gamma'\}$. Therefore, the reduction $c(|t|) \rightarrow_r \mathbb{T}_2$ (on a different redex than the one of $t \rightarrow_{\text{base}^r}^+ \mathbb{T}$) can only be performed either by reducing t' , or by reducing the ρ -redex $...\beta|\mu\gamma....|$, or if $t' = \mu\widetilde{\gamma}.\widetilde{\eta}|\widetilde{t}|$ and we reduce the ρ -redex $...\eta'|\mu\widetilde{\gamma}....|$. The first and second situation of the previous list have been already treated in the case $c = \xi$ (with the notation used there, it corresponds to the two diagrams of the subcase $t = \mu\gamma.\alpha|\mu\beta.\eta|s|$ and $\mathbb{T} = \mu\gamma.\eta|s|\{\alpha/\beta\}$). Also the third situation corresponds to the exact same case just mentioned, because the external $\mu\alpha.\beta|...|$ is not modified in neither reductions.

In the second case, then the reduction $c(|t|) \rightarrow_r \mathbb{T}_2$ can only be performed either by reducing $c''(|t|)$, or by reducing the ρ -redex $c(|t|)$. In the first situation the diagram follows easily by inductive hypothesis as in the previous case, and in the second one the diagram is the following:



thanks to Remark 2.23 and Lemma 2.25.

(4). Case $c = c'[\vec{u}]$.

Then $c(|t|) = c'(|t|)[\vec{u}]$ and the reduction $c(|t|) \rightarrow_r \mathbb{T}_2$ can only be performed either by reducing $c'(|t|)$, or reducing an element of $[\vec{u}]$, or the in case $c'(|t|)$ is a λ -abstraction or a μ -abstraction and we reduce the $\lambda\mu$ -redex $c(|t|)$.

In the first case one can easily use inductive hypothesis.

In the second case one can easily write the diagram.

In the third case $c'(|t|)$ is a λ -abstraction. Then either $c' = \xi$ and $t = \lambda x.t'$, or $c' = \lambda x.c''$. But the first case is impossible, because by hypothesis $t \rightarrow_{\text{base}^r}^+ \mathbb{T}$, so t cannot be a λ -abstraction; In the second case the diagram corresponds to the first diagram of the case 1 (with the notations used there, take $s := c''(|t|)$ and $\mathbb{S} := c''(|\mathbb{T}|)$).

In the fourth case $c'(|t|)$ is a μ -abstraction. Then either $c' = \xi$ and $t = \mu\alpha.\beta|t'|$, or $c' = \mu\alpha.\beta|c''|$. But in the first

situation, since by hypothesis $t \rightarrow_{\text{base}^r}^+ \mathbb{T}$, it must be $t = \mu\alpha.\beta|\mu\gamma.\eta|t''|$ and $\mathbb{T} = \mu\alpha.\eta|t''|\{\beta/\gamma\}$, and this situation has already been treated in the case $c = \xi$ (with the notations used there, it corresponds to the subcase $t = (\mu\alpha.\beta|s|)[\vec{u}]$ and $\mathbb{T} = \mu\alpha.\langle\beta|s|\rangle_{\alpha}^+[\vec{u}]$, where we consider the reduction of the ρ -redex¹¹). In the second case the diagram corresponds to the third diagram of the case 1 (with the notations used there, take $s := c''(|t|)$ and $\mathbb{S} := c''(|\mathbb{T}|)$).

(5). Case $c = v[c', \vec{u}]$.

Then $c(|t|) = v[c'(|t|), \vec{u}]$ and the reduction $c(|t|) \rightarrow_r \mathbb{T}_2$ can only be performed either by reducing $c'(|t|)$, or reducing v , or reducing an element of $[\vec{u}]$, or in the case v is a λ -abstraction or a μ -abstraction and we reduce the $\lambda\mu$ -redex $c(|t|)$.

In the first case one can trivially use the inductive hypothesis as done in the previous case.

In the second and third case one can easily write the diagram.

Let us look at the fourth case.

If v is a λ -abstraction, say $v = \lambda x.v'$, then the diagram corresponds to the second diagram of the case 1 (with the notations used there, take $w := c'(|t|)$ and $\mathbb{W} := c'(|\mathbb{T}|)$).

If v is a μ -abstraction, say $v = \mu\alpha.\beta|v'|$, then the diagram corresponds to the fourth diagram of the case 1 (with the notations used there, take $w := c'(|t|)$ and $\mathbb{W} := c'(|\mathbb{T}|)$). \square

A.3 APPENDIX - SECTION 3.1

We give proofs of Lemma 3.3, Proposition 3.4, Lemma 3.5, Proposition 3.7, Corollary 3.8.

Proof of Lemma 3.3. (1). Straightforward induction on M .

(2). Induction on M . Nothing changes w.r.t. the proof one does in λ -calculus, the only new case is $M = \mu\beta.\alpha|P|$ but it is done straightforwardly exactly as the case $M = \lambda x.P$.

(3). Induction on M .

Case $M = x$:

$$\begin{aligned}
 \mathcal{T}((M)_{\alpha}N) &= \mathcal{T}((x)_{\alpha}N) \\
 &= \{x\} \\
 &= \langle x \rangle_{\alpha}1 \\
 &= \bigcup_{\vec{u} \in !\mathcal{T}(N)} \langle x \rangle_{\alpha}[\vec{u}] \\
 &= \bigcup_{t \in \mathcal{T}(M), \vec{u} \in !\mathcal{T}(N)} \langle t \rangle_{\alpha}[\vec{u}].
 \end{aligned}$$

Case $M = \lambda x.P$:

$$\begin{aligned}
 \mathcal{T}((M)_{\alpha}N) &= \mathcal{T}(\lambda x.(P)_{\alpha}N) \\
 &= \{\lambda x.s \mid s \in \mathcal{T}((P)_{\alpha}N)\} \\
 &= \bigcup_{p \in \mathcal{T}(P)} \bigcup_{[\vec{u}] \in !\mathcal{T}(N)} \lambda x.(\langle p \rangle_{\alpha}[\vec{u}]) \\
 &= \bigcup_{p \in \mathcal{T}(P)} \bigcup_{[\vec{u}] \in !\mathcal{T}(N)} \langle \lambda x.p \rangle_{\alpha}[\vec{u}] \\
 &= \bigcup_{t \in \mathcal{T}(M)} \bigcup_{[\vec{u}] \in !\mathcal{T}(N)} \langle t \rangle_{\alpha}[\vec{u}].
 \end{aligned}$$

¹¹The diagrams are the last four of that subcase.

Case $M = \mu\beta.\gamma|P|$ (with $\beta, \gamma \neq \alpha$):

$$\begin{aligned} \mathcal{T}((M)_\alpha N) &= \mathcal{T}(\mu\beta.\gamma|(P)_\alpha N|) \\ &= \{\mu\beta.\gamma|s| \mid s \in \mathcal{T}((P)_\alpha N)\} \\ &= \bigcup_{p \in \mathcal{T}(P)} \bigcup_{[\vec{u}] \in !\mathcal{T}(N)} \mu\beta.\gamma|\langle p \rangle_\alpha[\vec{u}]| \\ &= \bigcup_{p \in \mathcal{T}(P)} \bigcup_{[\vec{u}] \in !\mathcal{T}(N)} \langle \mu\beta.\gamma|p| \rangle_\alpha[\vec{u}] \\ &= \bigcup_{t \in \mathcal{T}(M)} \bigcup_{[\vec{u}] \in !\mathcal{T}(N)} \langle t \rangle_\alpha[\vec{u}]. \end{aligned}$$

Case $M = \mu\beta.\alpha|P|$ (with $\beta \neq \alpha$):

$$\begin{aligned} \mathcal{T}((M)_\alpha N) &= \mathcal{T}(\mu\beta.\alpha|((P)_\alpha N)|) \\ &= \left\{ \mu\beta.\alpha|v[\vec{w}]| \mid [\vec{w}] \in !\mathcal{T}(N), \right. \\ &\quad \left. v \in \bigcup_{p \in \mathcal{T}(P), \vec{q} \in !\mathcal{T}(N)} \langle p \rangle_\alpha[\vec{q}] \right\} \\ &= \left\{ \mu\beta.\alpha|v[\vec{w}]| \mid [\vec{w}] \in !\mathcal{T}(N), v \in \langle p \rangle_\alpha[\vec{q}], \right. \\ &\quad \left. p \in \mathcal{T}(P), [\vec{q}] \in !\mathcal{T}(N) \right\} \\ &= \bigcup_{p \in \mathcal{T}(P)} \bigcup_{[\vec{u}] \in !\mathcal{T}(N)} \mu\beta. \sum_{\substack{([\vec{w}], [\vec{q}]) \\ \text{w.c. of } [\vec{u}]}} \alpha|\langle p \rangle_\alpha[\vec{q}]|[\vec{w}] \\ &= \bigcup_{p \in \mathcal{T}(P)} \bigcup_{[\vec{u}] \in !\mathcal{T}(N)} \mu\beta.\langle \alpha|p| \rangle_\alpha[\vec{u}] \\ &= \bigcup_{p \in \mathcal{T}(P)} \bigcup_{[\vec{u}] \in !\mathcal{T}(N)} \langle \mu\beta.\alpha|p| \rangle_\alpha[\vec{u}] \\ &= \bigcup_{t \in \mathcal{T}(M)} \bigcup_{[\vec{u}] \in !\mathcal{T}(N)} \langle t \rangle_\alpha[\vec{u}]. \end{aligned}$$

Case $M = PQ$:

$$\begin{aligned} \mathcal{T}((M)_\alpha N) &= \mathcal{T}(((P)_\alpha N)((Q)_\alpha N)) \\ &= \bigcup_{n \in \mathbb{N}} \bigcup_{p \in \mathcal{T}(P)} \bigcup_{[\vec{q}] \in !\mathcal{T}(Q)} \bigcup_{[\vec{s}^0], \dots, [\vec{s}^n] \in !\mathcal{T}(N)} \\ &\quad \left\{ v[w_1, \dots, w_n] \mid v \in \langle p \rangle_\alpha[\vec{s}^0], w_i \in \langle q_i \rangle_\alpha[\vec{s}^i] \right\} \\ &= \bigcup_{n \in \mathbb{N}} \bigcup_{p \in \mathcal{T}(P)} \bigcup_{[\vec{q}] \in !\mathcal{T}(Q)} \bigcup_{[\vec{u}] \in !\mathcal{T}(N)} \\ &\quad \sum_{\substack{([\vec{s}^0], \dots, [\vec{s}^n]) \\ \text{w.c. of } [\vec{u}]}} (\langle p \rangle_\alpha[\vec{s}^0])[\dots, \langle q_i \rangle_\alpha[\vec{s}^i], \dots] \\ &= \bigcup_{p \in \mathcal{T}(P)} \bigcup_{[\vec{q}] \in !\mathcal{T}(Q)} \bigcup_{[\vec{u}] \in !\mathcal{T}(N)} \langle p[\vec{q}] \rangle_\alpha[\vec{u}] \\ &= \bigcup_{t \in \mathcal{T}(M)} \bigcup_{\vec{u} \in !\mathcal{T}(N)} \langle t \rangle_\alpha[\vec{u}]. \end{aligned}$$

□

Proof of Proposition 3.4. For the first part of the proof, we have three subcases, corresponding to the three base-cases of the reduction.

Subcase $M = (\mu\alpha.\beta|P|)Q$, $N = \mu\alpha.\langle \beta|P| \rangle_\alpha Q$.

(1). If $s \in \mathcal{T}((\mu\alpha.\beta|P|)Q)$ then $s = (\mu\alpha.\beta|p|)[\vec{q}]$ for $p \in \mathcal{T}(P)$ and $[\vec{q}] \in !\mathcal{T}(Q)$. So $s \rightarrow_r \mu\alpha.\langle \beta|p| \rangle_\alpha[\vec{q}] \subseteq \mathcal{T}(\mu\alpha.\langle \beta|P| \rangle_\alpha Q)$ thanks to Lemma 3.3.

(2). If $s' \in \mathcal{T}(\mu\alpha.\langle \beta|P| \rangle_\alpha Q)$ then thanks to Lemma 3.3, $s' \in \mu\alpha.\langle \beta|p| \rangle_\alpha[\vec{q}]$ for $p \in \mathcal{T}(P)$ and $[\vec{q}] \in !\mathcal{T}(Q)$. So $\mathcal{T}((\mu\alpha.\beta|P|)Q) \ni (\mu\alpha.\beta|p|)[\vec{q}] \rightarrow_r \mu\alpha.\langle \beta|p| \rangle_\alpha[\vec{q}] \ni s'$, and $\mu\alpha.\langle \beta|p| \rangle_\alpha[\vec{q}] \subseteq \mathcal{T}(\mu\alpha.\langle \beta|P| \rangle_\alpha Q)$.

Subcase $M = \mu\gamma.\alpha|\mu\beta.\eta|P|$, $N = \mu\gamma.\eta|P|\{\alpha/\beta\}$. (1) and (2) are straightforward using Lemma 3.3 as above.

Subcase $M = (\lambda x.P)Q$, $N = P\{Q/x\}$. (1) and (2) are straightforward using Lemma 3.3 as above.

The “furthermore” is by induction on the single-hole context C s.t. $M = C(|M'|)$, $N = C(|N'|)$ and $M' \rightarrow_{\text{base}} N'$ (such a C exists because \rightarrow is contextual). The base case $C = \xi$ coincides with the above case. The step cases are straightforwardly done as one does in λ -calculus, treating the case of μ -abstraction exactly as a λ -abstraction. □

Proof of Lemma 3.5. Such a statement may seem strange at first sight, because one would expect (3) (which is the item used in the proof of Theorem 3.6, together with (1)) to be an inductive step of (2). However, (3) is *not* an inductive step of (2), simply because $\eta|p| \notin \lambda\mu^r$. This is due to the fact that we are in $\lambda\mu$ -calculus and not in Saurin’s $\Lambda\mu$ -calculus. One could be then tempted to state it in $\Lambda\mu$ -calculus, so only with item (2). In this case, since p would be in $\Lambda\mu$ and not just in $\lambda\mu$, (3) would in fact be an inductive step of (2), but still this is not what we need: in fact when we use (3) in the present paper, we want p to be in $\lambda\mu$, something which is not guaranteed by the statement in $\Lambda\mu$. Let us now prove the Lemma.

(1). Induction on P , similar to as it is done in [11].

(2). Induction on P . Let us see the case $P = \mu\alpha.\beta|M|$, the other cases being similar. Then $p = \mu\alpha.\beta|s|$ and $p' = \mu\alpha.\beta|s'|$, with $s, s' \in \mathcal{T}(M)$. Let $h \in \langle p \rangle_\gamma[\vec{d}] \cap \langle p' \rangle_\gamma[\vec{d}']$. Choosing $\alpha \neq \gamma$, we have two subcases.

Subcase $\beta \neq \gamma$: then $h = \mu\alpha.\beta|h_0|$ with $h_0 \in \langle s \rangle_\gamma[\vec{d}] \cap \langle s' \rangle_\gamma[\vec{d}']$. We can easily conclude by inductive hypothesis.

Subcase $\beta = \gamma$: then $h = \mu\alpha.\gamma|h_0[\vec{v}]|$, with $h_0 \in \langle s \rangle_\gamma[\vec{w}] \cap \langle s' \rangle_\gamma[\vec{w}']$ and where $([\vec{v}], [\vec{w}])$ and $([\vec{v}], [\vec{w}'])$ are w.c. respectively of $[\vec{d}]$ and of $[\vec{d}']$. Thus by inductive hypothesis we have $s = s'$, i.e. $p = p'$, and also $[\vec{w}] = [\vec{w}']$. Finally, $[\vec{d}] = [\vec{w}] * [\vec{v}] = [\vec{w}'] * [\vec{v}] = [\vec{d}']$.

(3). Let $\alpha \neq \gamma$ and fresh. Then $\langle \eta|p| \rangle_\gamma[\vec{d}] \cap \langle \eta|p'| \rangle_\gamma[\vec{d}'] \neq \emptyset$ iff $\mu\alpha.\langle \eta|p| \rangle_\gamma[\vec{d}] \cap \mu\alpha.\langle \eta|p'| \rangle_\gamma[\vec{d}'] \neq \emptyset$. The latter condition is $\langle \mu\alpha.\eta|p| \rangle_\gamma[\vec{d}] \cap \langle \mu\alpha.\eta|p'| \rangle_\gamma[\vec{d}']$, so it is of the shape considered by (2), and $\mu\alpha.\eta|p|, \mu\alpha.\eta|p'| \in \lambda\mu$. Therefore (2) gives $\mu\alpha.\eta|p| = \mu\alpha.\eta|p'|$, i.e. $p = p'$, as well as $[\vec{d}] = [\vec{d}']$. □

Proof of Proposition 3.7. We have the three base-case reductions:

Case $t = (\mu\alpha.\beta|p|)[\vec{q}]$ and $\mathbb{T}' = \mu\alpha.\langle \beta|p| \rangle_\alpha[\vec{q}]$. So it must be $M = (\mu\alpha.\beta|P|)Q$ with $p \in \mathcal{T}(P)$ and $[\vec{q}] \in !\mathcal{T}(Q)$. Now thanks to Lemma 3.3 we can take $N := \mu\alpha.\langle \beta|P| \rangle_\alpha Q$.

Case $t = (\lambda x.p)[\vec{q}]$ and $\mathbb{T}' = p([\vec{q}]/x)$, or case $t = \mu\gamma.\alpha|\mu\beta.\eta|p|$ and $\mathbb{T}' = \mu\gamma.\eta|p|\{\alpha/\beta\}$. Exactly as above, thanks to Lemma 3.3. \square

We now turn to Corollary 3.8. It relies on a result that we did not report in the main paper: Proposition A.18, which appears in the Appendix A.5. It is the generalisation of Proposition 3.7 mentioned in the proof of Corollary 3.8 in the main paper. This result requires some discussion, so we postpone its proof and dedicate the whole Appendix A.5. to it. In the meantime, below, we prove Corollary 3.8 using Proposition A.18.

Proof of Corollary 3.8. We prove, by induction on $\ell \in \mathbb{N}$, that if ℓ is the length of a maximal reduction (let's call it ϕ) from a sum \mathbb{T} to its normal form $\text{nf}_r(\mathbb{T})$, then the statement of Corollary 3.8 holds. If $\ell = 0$ then just take $N := M$. If $\ell \geq 1$, then ϕ factorizes as $\mathbb{T} \rightarrow_r \mathbb{T}' \rightarrow_r \text{nf}_r(\mathbb{T})$ for some sum \mathbb{T}' . Since $\mathbb{T} \in \mathcal{T}(M)$ by hypothesis, by Proposition A.18 there exist $N' \in \lambda\mu$ and $\mathbb{T}'' \subseteq \mathcal{T}(N')$ such that $M \rightarrow N'$ and $\mathbb{T}' \rightarrow_r \mathbb{T}''$. Let $k \geq 0$ be the length of this last reduction. By confluence, we have also $\mathbb{T}'' \rightarrow_r \text{nf}_r(\mathbb{T})$. Now take the maximal reduction from \mathbb{T}'' to $\text{nf}_r(\mathbb{T}'') = \text{nf}_r(\mathbb{T})$ and let ℓ' be its length. Due to the maximality of ϕ , it must be $\ell' + k + 1 \leq \ell$, so $\ell' < \ell$, and now we can apply the inductive hypothesis to ℓ' (because it is the maximal length of a reduction between a sum \mathbb{T}'' and its normal form). Since we already found that $\mathbb{T}'' \subseteq \mathcal{T}(N')$, we get an $N \in \lambda\mu$ such that $M \rightarrow N' \rightarrow N$ and $\text{nf}_r(\mathbb{T}) = \text{nf}_r(\mathbb{T}'') \subseteq \mathcal{T}(N)$. \square

A.4 APPENDIX - SECTION 3.2

We give proofs of Lemma 3.12 and of Lemma 3.13.

Proof of Lemma 3.12. If t is a hnf, we can easily conclude: since any eventual bag of s is empty, and since reductions cannot erase non-empty bags (because *linear*), the fact that $s \in \text{nf}_r(t)$ entails that already t contains only empty bags. But in a hnf the reduction can only take place inside some bag, so it actually must be $s = t$. Therefore, $s \in s = t = H^0(t)$ and we are done taking $n := 0$.

So we are left with the case in which t is *not* hnf. In this case we know that $t \rightarrow_r H(t)$. By confluence, $H(t) \rightarrow_r \text{nf}_r(t)$. But since $s \in \text{nf}_r(t)$, there is some $t_1 \in H(t)$ s.t. $s \in \text{nf}_r(t_1)$. Now we can reason as in the beginning, splitting in two cases: either t_1 is a hnf, in which case we reason exactly as in the first four lines of the proof: by linearity we get $s = t_1 \in H(t)$, and we are done taking $n := 1$. Or t_1 is not a hnf. In this case we can reason again as before, obtaining a $t_1 \rightarrow_r H(t_1)$, $H(t_1) \rightarrow_r \text{nf}_r(t_1)$ and a $t_2 \in H(t_1)$ s.t. $s \in \text{nf}_r(t_2)$. We can keep going with the same splits: if t_2 is hnf, we are done taking $n := 2$; if t_2 is not hnf, we obtain a new t_3 as before. Now, the generation of such a new t_{i+1} from the previously generated t_i cannot continue forever: we claim that there must be some $m \in \mathbb{N}$ for which t_m is hnf. If this is the

case, the proof is concluded because we can take $n := m$ as already mentioned. To see that such an m does exist, one simply remarks that at each time we have $t_i \rightarrow_r H(t_i)$ and $t_{i+1} \in H(t_i)$. But the well-founded measure $m(\cdot)$ is strictly decreasing along reductions, which precisely means that we obtain the strictly decreasing sequence:

$$m(t) > m(t_1) > m(t_2) > \dots$$

Therefore, it must terminate (because the order is well-founded) on some $m(t_m)$, for some $m \in \mathbb{N}$, and we are done as already explained. \square

The previous proof could clearly be given in an inductive way, but we chose to give it in this way to (maybe) let the reader better see the reasoning.

Proof of Lemma 3.13. We have to show that $\mathcal{T}(H(M)) = \bigcup_{t \in \mathcal{T}(M)} H(t)$.

By Lemma 1.4 we know that:

$$M = \lambda\vec{x}_1.\mu\alpha_1.\beta_1 | \dots | \lambda\vec{x}_k.\mu\alpha_k.\beta_k | RQ_1 \dots Q_n |$$

with the condition that either there is a ρ -redex in the head of M , or there is no such ρ -redex and R is *not* a variable (thus R is either a λ -redex or a μ -redex). We have just said in a different fashion that M is *not* a hnf. Now let us show the two inclusions.

(\subseteq). Take $s \in \mathcal{T}(H(M))$. We have three cases:

Case in which there is a ρ -redex in the head of M . Therefore there is also a leftmost ρ -redex $\dots \beta_i |\mu\alpha_{i+1} \dots |$ in the head of M . Then $H(M)$ is the term:

$$\lambda\vec{x}_1.\mu\alpha_1.\beta_1 | \dots | \lambda\vec{x}_i.\mu\alpha_i.\beta_{i+1} | \dots | \lambda\vec{x}_k.\mu\alpha_k.\beta_k | R\vec{Q} || \{\beta_i/\alpha_{i+1}\} |.$$

So s is the term:

$$\lambda\vec{x}_1.\mu\alpha_1.\beta_1 | \dots | \lambda\vec{x}_i.\mu\alpha_i.\beta_{i+1} | \dots | \lambda\vec{x}_k.\mu\alpha_k.\beta_k | r[\vec{q}^1] \dots [\vec{q}^n] || \{\beta_i/\alpha_{i+1}\} |$$

for $r \in \mathcal{T}(R)$ and $[\vec{q}^i] \in !\mathcal{T}(Q_i)$. But $\mathcal{T}(M)$ contains the element:

$$\lambda\vec{x}_1.\mu\alpha_1.\beta_1 | \dots | \lambda\vec{x}_k.\mu\alpha_k.\beta_k | r[\vec{q}^1] \dots [\vec{q}^n] || =: t$$

and thus $s = H(t)$, i.e. $s \in \bigcup_{t \in \mathcal{T}(M)} H(t)$.

Case there are no ρ -redexes and $R = (\mu\gamma.\eta|P|)D$. Then $H(M) = \vec{\lambda}\mu. | (\mu\gamma.(\eta|P|)_\gamma D)\vec{Q} |$. So by Lemma 3.3 $s \in \vec{\lambda}\mu. | (\mu\gamma.\langle \eta | p | \rangle_\gamma [\vec{d}]) [\vec{q}^1] \dots [\vec{q}^n] |$ for a $p \in \mathcal{T}(P)$, $[\vec{d}] \in !\mathcal{T}(D)$ and $[\vec{q}^i] \in !\mathcal{T}(Q_i)$. So $s \in H(t)$, with t begin the term:

$$\lambda\vec{x}_1.\mu\alpha_1.\beta_1 | \dots | \lambda\vec{x}_k.\mu\alpha_k.\beta_k | (\mu\gamma.\eta|p|) [\vec{d}] [\vec{q}^1] \dots [\vec{q}^n] ||$$

which belongs to $\mathcal{T}(M)$, i.e. $s \in \bigcup_{t \in \mathcal{T}(M)} H(t)$.

Case there are no ρ -redexes and $R = (\lambda x.P)D$. Exactly as above using Lemma 3.3.

(\supseteq). One can follow the exact same kind of argument as before: the fact that Taylor expansion preserves the structure of the term, plus Lemma 3.3, is what makes us able to transport one step of the head reduction from terms to resource terms. \square

A.5 APPENDIX - Rigids associated with a resource term

This appendix section does not correspond to a section in the main paper. Its aim is to prove Proposition A.18, a result that is needed in the proof of Corollary 3.8, as previously explained in Appendix A.3.

We report here a proof involving the “rigid resource terms”: words built in the same way as resource terms but taking lists instead of bags (multiset of terms).

We present these constructions in detail also because they are needed in the proof of Stability (Theorem 4.1), which is done exactly as in [1, Theorem 5.11], and whose detailed proof is given in the next Appendix A.6 for the seek of completeness; in particular, the following Lemma A.16 and Lemma A.17, involving these constructions, are used in the proof of Theorem 4.1.

Definition A.12. *The set of rigid terms is defined by:*

$$t ::= x \mid \lambda x.t \mid \mu\alpha.\beta|t| \mid t \langle \dots, t \rangle \quad (\text{here } \langle \dots \rangle \text{ means a list}).$$

The set of rigid k -context is defined as expected adding the clause “ $\xi_1 \mid \dots \mid \xi_k$ ” for the holes.

Definition A.13. *Let c be a resource- k -context. We define a set $\text{Rigid}(c)$ of rigid k -contexts, whose elements are called the rigids of c , by induction on c as follows:*

1. $\text{Rigid}(\xi_i) = \{\xi_i\}$
2. $\text{Rigid}(x) = \{x\}$
3. $\text{Rigid}(\lambda x.c_0) = \{\lambda x.c_0^* \mid c_0^* \in \text{Rigid}(c_0)\}$
4. $\text{Rigid}(\mu\alpha.\beta|c_0|) = \{\mu\alpha.\beta|c_0^*| \mid c_0^* \in \text{Rigid}(c_0)\}$
5. $\text{Rigid}(c_0[c_1, \dots, c_k]) = \{c_0^* \langle c_{\sigma(1)}^*, \dots, c_{\sigma(k)}^* \rangle \mid c_i^* \in \text{Rigid}(c_i) \text{ and } \sigma \text{ a permutation on } k \text{ elements}\}$

The rigids of a resource k -context c are the rigid k -contexts which can be “canonically associated” with c . Rigid-resource-calculus has been considered, e.g., in [24] and its study sheds light on the combinatorial role of the factorial coefficients in the full (i.e. quantitative) Taylor expansion of a term. Our set $\text{Rigid}(c)$ is the preimage $F^{-1}(c)$ of c under the surjection F^{12} from rigid contexts to resource context simply forgetting the order of the lists. Its graph is what, in [24], is called the “representation relation”. In the following definition we precisely operate such a forgetful operation, but in addition we consider terms filling the holes.

Definition A.14. *Let c^* be a rigid of a resource- k -context c and, for $i = 1, \dots, k$, let $\vec{v}^i := \langle v_1^i, \dots, v_{\deg_{\xi_i}(c)}^i \rangle$ be a list¹³ of resource terms. We define, by induction on c , a resource term $c^*(\vec{v}^1, \dots, \vec{v}^k)$ as follows:*

1. If $c = \xi_i$ then $c^* = \xi_i$; we set $c^*(\langle \rangle, \dots, \langle \rangle, \langle v_1^i \rangle, \langle \rangle, \dots, \langle \rangle)$ to be v_1^i .
2. If $c = x$ then $c^* = x$; we set $c^*(\langle \rangle, \dots, \langle \rangle) := x$.
3. If $c = \lambda x.c_0$ then $c^* = \lambda x.c_0^*$ where c_0^* is a rigid of c_0 ; we set $c^*(\vec{v}^1, \dots, \vec{v}^k) = \lambda x.c_0^*(\vec{v}^1, \dots, \vec{v}^k)$.

¹²In [24] F is called $\parallel \cdot \parallel$, but here write F , for “forgetful”.

¹³If $\deg_{\xi_i}(c) = 0$ we mean the empty list.

4. If $c = \mu\alpha.\beta|c_0|$ then $c^* = \mu\alpha.\beta|c_0^*|$ where c_0^* is a rigid of c_0 ; we set $c^*(\vec{v}^1, \dots, \vec{v}^k) = \mu\alpha.\beta|c_0^*(\vec{v}^1, \dots, \vec{v}^k)|$.
5. If $c = c_0[c_1, \dots, c_n]$, then $c^* = c_0^* \langle c_{\sigma(1)}^*, \dots, c_{\sigma(n)}^* \rangle$ where c_i^* is a rigid of c_i . So each list \vec{v}^i factorizes as a concatenation $\vec{w}^{i0} \vec{w}^{i1} \dots \vec{w}^{in}$ of lists where \vec{w}^{ij} has exactly $\deg_{\xi_i}(c_j)$ elements¹⁴; we set $c^*(\vec{v}^1, \dots, \vec{v}^k)$ to be the term:

$$c_0^* \langle \vec{w}^{10}, \dots, \vec{w}^{k0} \rangle [c_{\sigma(1)}^* \langle \vec{w}^{11}, \dots, \vec{w}^{k1} \rangle, \dots, c_{\sigma(n)}^* \langle \vec{w}^{1n}, \dots, \vec{w}^{kn} \rangle].$$

Remark A.15. *One clearly has that if $v \rightarrow_r \forall$ then:*

$$c^*(\langle \dots, \langle \dots, v, \dots \rangle, \dots \rangle)$$

\rightarrow_r -reduces to:

$$\sum_{w \in \forall} c^*(\langle \dots, \langle \dots, w, \dots \rangle, \dots \rangle) =: c^*(\langle \dots, \langle \dots, \forall, \dots \rangle, \dots \rangle).$$

Let us extend the definition of Taylor expansion to resource k -contexts by adding, in its definition, the clause:

$$\mathcal{T}(\xi_i) := \{\xi_i\}.$$

It is clear that if C is a k -context then all elements of $\mathcal{T}(C)$ are resource k -contexts.

In the following, if \vec{v} is a list, we denote with $[\vec{v}]$ the multiset associated with \vec{v} (same elements but disordered).

Lemma A.16. *Let C be a k -context and $c_1, c_2 \in \mathcal{T}(C)$. Let c_1^* and c_2^* rigids respectively of c_1 and c_2 . For $i = 1 \dots k$, let $\vec{v}^i = \langle v_1^i, \dots, v_{\deg_{\xi_i}(c_1)}^i \rangle$ and $\vec{u}^i = \langle u_1^i, \dots, u_{\deg_{\xi_i}(c_2)}^i \rangle$ be lists of resource terms. If $c_1^*(\vec{v}^1, \dots, \vec{v}^k) = c_2^*(\vec{u}^1, \dots, \vec{u}^k)$ then $c_1 = c_2$ and $[\vec{v}^i] = [\vec{u}^i]$ for all i .*

Proof. Induction on C .

Case $C = \xi_i$. Then $c_1 = \xi_i = c_2$, and $\vec{v}^i = \langle v^i \rangle$, $\vec{u}^i = \langle u^i \rangle$ and $\vec{v}^j = \langle \rangle = \vec{u}^j$ for $j \neq i$. So $v^i = c_1^*(\vec{v}^1, \dots, \vec{v}^k) = c_2^*(\vec{u}^1, \dots, \vec{u}^k) = u^i$.

Case $C = x$. Trivial.

Case $C = \lambda x.C_0$ and case $C = \mu\alpha.\beta|C_0|$. Trivial by inductive hypothesis.

Case $C = C' C''$. Then, for $i = 1, 2$, one has $c_i = c_{i0}[c_{i1}, \dots, c_{in_i}]$ with $c_{i0} \in \mathcal{T}(C')$ and $c_{ij} \in \mathcal{T}(C'')$ for $j \geq 1$. So $c_i^* = c_{i0}^* \langle c_{i\sigma_i(1)}^*, \dots, \lambda x.c_{i\sigma_i(n_i)}^* \rangle$ where σ_i is a permutation on n_i elements. So $c_1^*(\vec{v}^1, \dots, \vec{v}^k)$ is the term:

$$c_{10}^* \langle \vec{w}^{110}, \dots, \vec{w}^{1k0} \rangle [c_{1\sigma_1(1)}^* \langle \vec{w}^{111}, \dots, \vec{w}^{1k1} \rangle, \dots, c_{1\sigma_1(n_1)}^* \langle \vec{w}^{11n_1}, \dots, \vec{w}^{1kn_1} \rangle]$$

and $c_2^*(\vec{v}^1, \dots, \vec{v}^k)$ is the term:

$$c_{20}^* \langle \vec{w}^{210}, \dots, \vec{w}^{2k0} \rangle [c_{2\sigma_2(1)}^* \langle \vec{w}^{211}, \dots, \vec{w}^{2k1} \rangle, \dots, c_{2\sigma_2(n_2)}^* \langle \vec{w}^{21n_2}, \dots, \vec{w}^{2kn_2} \rangle]$$

¹⁴This is a concise way of saying that we take $\vec{w}^{i1} := \langle v_1^i, \dots, v_{\deg_{\xi_i}(c_1)}^i \rangle$, $\vec{w}^{i2} := \langle v_{1+\deg_{\xi_i}(c_1)}^i, \dots, v_{\deg_{\xi_i}(c_2)+\deg_{\xi_i}(c_1)}^i \rangle$ etc.

where the concatenation $\vec{w}^{1j_1} \dots \vec{w}^{1j_{n_1}}$ gives \vec{v}^j and the concatenation $\vec{w}^{2j_1} \dots \vec{w}^{2j_{n_2}}$ gives \vec{u}^j . From $c_1^\bullet(\vec{v}^1, \dots, \vec{v}^k) = c_2^\bullet(\vec{u}^1, \dots, \vec{u}^k)$ we get that: $n_1 = n_2 =: n$, that:

$$c_{10}^\bullet(\vec{w}^{110}, \dots, \vec{w}^{1k0}) = c_{20}^\bullet(\vec{w}^{210}, \dots, \vec{w}^{2k0}) \quad (2)$$

and that there exist a permutation ρ on n elements which identifies each term of the writte bag of $c_1^\bullet(\vec{v}^1, \dots, \vec{v}^k)$ with the respective one of the written bag of $c_2^\bullet(\vec{u}^1, \dots, \vec{u}^k)$. That is, for all $h = 1, \dots, n$, one has:

$$c_{1\sigma_1(h)}^\bullet(\vec{w}^{11h}, \dots, \vec{w}^{1kh}) = c_{2\sigma_2(\rho(h))}^\bullet(\vec{w}^{21\rho(h)}, \dots, \vec{w}^{2k\rho(h)}) \quad (3)$$

Now the inductive hypothesis on (2) gives $c_{10}^\bullet = c_{20}^\bullet$ as well as $\vec{w}^{1i0} = \vec{w}^{2i0}$, and the inductive hypothesis on (3) gives, at the end of the day, $[c_{11}^\bullet, \dots, c_{1n}^\bullet] = [c_{21}^\bullet, \dots, c_{2n}^\bullet]$ as well as $[\vec{w}^{1j_1} \dots \vec{w}^{1j_n}] = [\vec{w}^{2j_1} \dots \vec{w}^{2j_n}]$ for $j \geq 1$. Putting these things together, we have the desired result. \square

Lemma A.17. 1. Let C be a k -context.

Then $\mathcal{T}(C(M_1, \dots, M_k))$ is the set:

$$\{c^\bullet(\vec{s}^1, \dots, \vec{s}^k) \mid c \in \mathcal{T}(C), c^\bullet \text{ rigid of } c \text{ and } \vec{s}^i \text{ list of elements of } \mathcal{T}(M_i)\}.$$

2. Let $c = c(\xi)$ single-hole resource context, $M \in \lambda\mu$ and $s_0 \in \lambda\mu^r$. If $c(s_0) \in \mathcal{T}(M)$, then there is a context $C = C(\xi)$, an $N \in \lambda\mu$, a resource context $\tilde{c} \in \mathcal{T}(C)$, a rigid \tilde{c}^\bullet of c and $s_1, \dots, s_{\deg_\xi \tilde{c} - 1} \in \mathcal{T}(N)$ s.t.

- $M = C(N)$
- $s_0 \in \mathcal{T}(N)$
- $c(t) = \tilde{c}^\bullet(\langle t, s_1, \dots, s_{\deg_\xi \tilde{c} - 1} \rangle)$ for all $t \in \lambda\mu^r$.

Proof. (1). Straightforward induction on C .

(2). Induction on the single-hole resource context c .

Case $c = \xi$. Take $C := \xi$, $N := M$, $\tilde{c} := \xi$, $\tilde{c}^\bullet := \xi$ (there are no s_i 's because $\deg_\xi \tilde{c} - 1 = 0$).

Case $c = \mu\alpha.\beta|c_1|$. Since $c(s_0) \in \mathcal{T}(M)$ then $M = \mu\alpha.\beta|M_1|$ with $c_1(s_0) \in \mathcal{T}(M_1)$. We can thus take $C := \mu\alpha.\beta|C_1|$, $\tilde{c} := \mu\alpha.\beta|\tilde{c}_1|$, $\tilde{c}^\bullet := \mu\alpha.\beta|\tilde{c}_1^\bullet|$, where $C_1, N, \tilde{c}_1, \tilde{c}_1^\bullet$ and $s_1, \dots, s_{\deg_\xi \tilde{c}_1}$ are given by the inductive hypothesis.

Case $c = \lambda x.c_1$. Exactly as the case of μ -abstraction.

Case $c = c'[\vec{u}]$. Analogous as above.

Case $c = u[c', u_1, \dots, u_n]$. Since $u(c'(s_0), \vec{u}) = c(s_0) \in \mathcal{T}(M)$ then M must have shape $M = PQ$ with $u \in \mathcal{T}(P)$ and $c'(s_0), u_i \in \mathcal{T}(Q)$. By induction hypothesis, we can write $Q = C_0(N)$ for an appropriate context C_0 and $N \in \lambda\mu$ s.t. $s_0 \in \mathcal{T}(N)$, and also there is a resource context $c_0 \in \mathcal{T}(C_0)$ and c_0^\bullet a rigid of c_0 , together with a list $\vec{s}^0 := \langle s_1^0, \dots, s_{\deg_\xi(c_0) - 1}^0 \rangle$ of elements of $\mathcal{T}(N)$, such that $c'(t) = c_0^\bullet(\langle t, \vec{s}^0 \rangle)$ for all $t \in \lambda\mu^r$. But $u_i \in \mathcal{T}(Q) = \mathcal{T}(C_0(N))$, so by the point (1), each u_i have shape $u_i = c_i^\bullet(\vec{s}^i)$ for an appropriate c_i^\bullet rigid of some $c_i \in \mathcal{T}(C_0)$ and some \vec{s}^i list of elements of $\mathcal{T}(N)$. Now remark that $M = C(N)$ for $C := PC_0$. Moreover, putting $\tilde{c} := u[c_0, c_1, \dots, c_n] \in \mathcal{T}(C)$ and choosing its rigid $\tilde{c}^\bullet := u^\bullet \langle c_0^\bullet, c_1^\bullet, \dots, c_n^\bullet \rangle$ (which rigid u^\bullet of u one choses does not matter), we have: $c(t) = u[c'(t), u_1, \dots, u_n] =$

$u[c_0^\bullet(\langle t, \vec{s} \rangle), c_1^\bullet(\vec{s}^1), \dots, c_n^\bullet(\vec{s}^n)] = \tilde{c}^\bullet(\langle t, \vec{s} \rangle)$ for all $t \in \lambda^r$, where the list \vec{s} is the concatenation $\vec{s}^0 \vec{s}^1 \dots \vec{s}^n$ is a list of $\deg_\xi(\tilde{c}) - 1$ elements of $\mathcal{T}(N)$. \square

We can now finally prove the full Proposition A.18.

Proposition A.18. If $\mathcal{T}(M) \supseteq \mathbb{T} \rightarrow_r \mathbb{T}'$ then there is $N \in \lambda\mu$ and a sum $\tilde{\mathbb{T}} \subseteq \mathcal{T}(N)$ s.t. $M \rightarrow N$ and $\mathbb{T}' \rightarrow_r \tilde{\mathbb{T}}$.

Proof. Saying that $\mathbb{T} \rightarrow_r \mathbb{T}'$ means that \mathbb{T} has shape $\mathbb{T} = \sum_i t_i + c(\langle h \rangle)$ and \mathbb{T}' has shape $\mathbb{T}' = \sum_i t_i + c(\langle \mathcal{H} \rangle)$, for some single-hole resource context c , a resource term h and a sum \mathcal{H} s.t. $h \rightarrow_{\text{base}} \mathcal{H}$. But since $c(\langle h \rangle) \in \mathbb{T} \subseteq \mathcal{T}(M)$, by Lemma A.17(2) we get a context C_0 , a term $N' \in \lambda\mu$, a resource context $c_0 \in \mathcal{T}(C_0)$, a rigid c_0^\bullet of c_0 and resource terms $\vec{s} \in \mathcal{T}(N')$ s.t. $M = C_0(N')$, $h \in \mathcal{T}(N')$ and $c_0(u) = c_0^\bullet(u, \vec{s})$ for all $u \in \lambda\mu^r$. Now we can apply Proposition 3.7 to $h \in \mathcal{T}(N')$ obtaining an $N'' \in \lambda\mu$ s.t. $N' \rightarrow N''$ and $\mathcal{H} \subseteq \mathcal{T}(N'')$. Set $N := C_0(N'')$, so that $M = C_0(N') \rightarrow N$. Now: every $t_i \in \mathcal{T}(M) = \mathcal{T}(C_0(N'))$, so by Lemma A.17(1) it must have shape $t_i = c_i^\bullet(\vec{v}_i)$ for some resource terms $v_{ij} \in \mathcal{T}(N')$, a context $c_i \in \mathcal{T}(C_0)$ and a rigid c_i^\bullet of c_i . But since $N' \rightarrow N''$ we can apply Proposition 3.4(1) on v_{ij} and obtain that $v_{ij} \rightarrow_r \mathbb{V}_{ij}$ for some sum $\mathbb{V}_{ij} \subseteq \mathcal{T}(N'')$. So $t_i \rightarrow_r c_i^\bullet(\vec{\mathbb{V}}_i)$. Let's call $\mathbb{T}_i := c_i^\bullet(\vec{\mathbb{V}}_i)$. Using again Lemma A.17(1) one has that $\mathbb{T}_i \subseteq \mathcal{T}(N)$. Now, let's use again Proposition 3.4(1), this time on $s \in \mathcal{T}(N')$. Since $N' \rightarrow N''$ we obtain sums $\mathbb{S}_i \subseteq \mathcal{T}(N'')$ s.t. $s_i \rightarrow_r \mathbb{S}_i$. So we have: $c(\langle \mathcal{H} \rangle) = c_0^\bullet(\langle \mathcal{H}, \vec{s} \rangle) \rightarrow_r c_0^\bullet(\langle \mathcal{H}, \mathbb{S} \rangle) =: \mathbb{U}$. But since $\mathcal{H} \subseteq \mathcal{T}(N'')$ and every $\mathbb{S}_i \subseteq \mathcal{T}(N'')$, again thanks to Lemma A.17(1) one has $\mathbb{U} \subseteq \mathcal{T}(C_0(N'')) = \mathcal{T}(N)$. This ends the proof, since letting $\tilde{\mathbb{T}} := \sum_i \mathbb{T}_i + \mathbb{U} \subseteq \mathcal{T}(N)$ one has $\mathbb{T}' \rightarrow_r \tilde{\mathbb{T}}$. \square

A.6 APPENDIX - SECTION 4.1

We give the proof of Theorem 4.1. The proof is taken from [1].

Proof of Theorem 4.1. Since every X_i is \mathcal{T} -bounded, for $i = 1, \dots, n$ there exists $L_i \in \Lambda$ s.t. $\bigcup_{N \in X_i} \text{NFT}(N) \subseteq \text{NFT}(L_i)$. Fix now $M_1, \dots, M_n \in \Lambda$ s.t. $\text{NFT}(M_i) = \bigcap_{N \in X_i} \text{NFT}(N)$. We

have to show that:

$$\text{NFT}(C(M_1, \dots, M_n)) = \bigcap_{N_1 \in X_1} \dots \bigcap_{N_n \in X_n} \text{NFT}(C(N_1, \dots, N_n)).$$

(\subseteq). Clearly, for all $i = 1, \dots, n$ and $N_i \in X_i$, we have $\text{NFT}(M_i) \subseteq \text{NFT}(N_i)$, therefore we conclude that:

$$\text{NFT}(C(M_1, \dots, M_n)) \subseteq \text{NFT}(C(N_1, \dots, N_n))$$

by Monotonicity (Theorem 3.2).

(\supseteq). Let $t \in \bigcap_{\vec{N} \in \vec{X}} \text{NFT}(C(N_1, \dots, N_n))$ (where we put $\vec{N} \in \vec{X}$) $\vec{N} := (N_1, \dots, N_n)$ and $\vec{X} := (X_1, \dots, X_n)$. For every $\vec{N} \in \vec{X}$, by Lemma A.17 there exist an n -resource-context $c_{\vec{N}} \in \mathcal{T}(C)$ and, for every $i = 1, \dots, n$, a list $\vec{v}_{\vec{N}}^i = \langle v_{\vec{N}}^{i1}, \dots, v_{\vec{N}}^{id_i} \rangle$ (where $d_i := \deg_{\xi_i}(c_{\vec{N}})$) of elements of $\mathcal{T}(N_i)$ and such that $t \in$

$\text{nf}_r(c_{\vec{N}}^\bullet(\vec{v}_{\vec{N}}^1, \dots, \vec{v}_{\vec{N}}^n))$, for $c_{\vec{N}}^\bullet$ a rigid of $c_{\vec{N}}$. Fix any reduction from $c_{\vec{N}}^\bullet(\vec{v}_{\vec{N}}^1, \dots, \vec{v}_{\vec{N}}^n)$ to its normal form, and confluence allows to factorize it as follows:

$$\begin{aligned} & c_{\vec{N}}^\bullet(\text{nf}_r(v_{\vec{N}}^{11}), \dots, \text{nf}_r(v_{\vec{N}}^{1d_1}), \dots, \text{nf}_r(v_{\vec{N}}^{n1}), \dots, \text{nf}_r(v_{\vec{N}}^{nd_n})) \\ & \xrightarrow{r} \\ & \text{nf}_r(c_{\vec{N}}^\bullet(\vec{v}_{\vec{N}}^1, \dots, \vec{v}_{\vec{N}}^n)) \ni t. \end{aligned}$$

So for all $i = 1, \dots, n$ and $j = 1, \dots, d_i$, there exist $w_{\vec{N}}^{ij} \in \text{nf}_r(v_{\vec{N}}^{ij})$ such that:

$$\text{nf}_r(c_{\vec{N}}^\bullet(\vec{w}_{\vec{N}}^1, \dots, \vec{w}_{\vec{N}}^n)) \ni t \quad (4)$$

and being $N_i \in \mathcal{X}_i$ which is bounded by L_i , we have $w_{\vec{N}}^{ij} \in \text{nf}_r(v_{\vec{N}}^{ij}) \subseteq \text{NF}\mathcal{T}(N_i) \subseteq \text{NF}\mathcal{T}(L_i)$. From each inclusion $w_{\vec{N}}^{ij} \in \text{NF}\mathcal{T}(L_i)$ we obtain a resource term $u_{\vec{N}}^{ij} \in \mathcal{T}(L_i)$ such that:

$$w_{\vec{N}}^{ij} \in \text{nf}_r(u_{\vec{N}}^{ij}) \quad (5)$$

By composing thus a reduction from $u_{\vec{N}}^{ij}$ to $w_{\vec{N}}^{ij}$ with a reduction from $c_{\vec{N}}^\bullet(\vec{w}_{\vec{N}}^1, \dots, \vec{w}_{\vec{N}}^n)$ to t , we find that t belongs to $\text{nf}_r(c_{\vec{N}}^\bullet(\vec{u}_{\vec{N}}^1, \dots, \vec{u}_{\vec{N}}^n))$. This holds for all $\vec{N} \in \vec{\mathcal{X}}$, i.e.:

$$t \in \bigcap_{\vec{N} \in \vec{\mathcal{X}}} \text{nf}_r(c_{\vec{N}}^\bullet(\vec{u}_{\vec{N}}^1, \dots, \vec{u}_{\vec{N}}^n)). \quad (6)$$

Now, Lemma A.17 gives $c_{\vec{N}}^\bullet(\vec{u}_{\vec{N}}^1, \dots, \vec{u}_{\vec{N}}^n) \in \mathcal{T}(C(L_1, \dots, L_n))$. But since the L_i 's are independent from N_1, \dots, N_n , and thanks to (6), we can apply Lemma 3.6, and obtain that the set $\{c_{\vec{N}}^\bullet(\vec{u}_{\vec{N}}^1, \dots, \vec{u}_{\vec{N}}^n) \mid \vec{N} \in \vec{\mathcal{X}}\}$ is actually a singleton. Therefore, Lemma A.17(2) tells us that also the terms $c_{\vec{N}}$ and the bags $[\vec{u}_{\vec{N}}^i]$ are independent from $\vec{N} \in \vec{\mathcal{X}}$. The unique element of the previous singleton has hence shape $c^\bullet(\vec{u}^1, \dots, \vec{u}^n)$, with c^\bullet a rigid of a $c \in \mathcal{T}(C)$, and \vec{u}^i a list of elements of $\mathcal{T}(L_i)$. Recalling now that $\sum_j u_j^i \subseteq \mathcal{T}(L_i)$, we can apply Corollary 3.8 in order to obtain, for each $i = 1, \dots, n$, an $L'_i \in \Lambda$ s.t. $L_i \twoheadrightarrow L'_i$ and, using (5), $\sum_j w_j^{ij} \subseteq \mathcal{T}(L'_i)$. Thus Lemma A.17 tells us that, for every $\vec{N} \in \vec{\mathcal{X}}$, we have:

$$c^\bullet(\vec{w}_{\vec{N}}^1, \dots, \vec{w}_{\vec{N}}^n) \in \mathcal{T}(C(L'_1, \dots, L'_n)). \quad (7)$$

But now thanks to (7) and (4) (which holds for all $\vec{N} \in \vec{\mathcal{X}}$), we can apply again Lemma 3.6 in order to find that the set $\{c^\bullet(\vec{w}_{\vec{N}}^1, \dots, \vec{w}_{\vec{N}}^n) \mid \vec{N} \in \vec{\mathcal{X}}\}$ is a singleton. Again by Lemma A.17(2), we have that all the bags $[\vec{w}_{\vec{N}}^1], \dots, [\vec{w}_{\vec{N}}^n]$ for $\vec{N} \in \vec{\mathcal{X}}$, coincide respectively to some bags $[\vec{w}^1], \dots, [\vec{w}^n]$ which are independent from $\vec{N} \in \vec{\mathcal{X}}$. So the only element of the previous singleton has shape $c^\bullet(\vec{w}^1, \dots, \vec{w}^n)$, and (4) becomes:

$$t \in \text{nf}_r(c^\bullet(\vec{w}^1, \dots, \vec{w}^n)). \quad (8)$$

Now for all $i = 1, \dots, n$, we already know that $[\vec{w}^i] = [\vec{w}_{\vec{N}}^i]$ which is a bag of elements of $\text{NF}\mathcal{T}(N)$, and this holds for all $N \in \mathcal{X}_i$. That is, we have:

$$\sum_j \vec{w}^{ij} \subseteq \bigcap_{N \in \mathcal{X}_i} \text{NF}\mathcal{T}(N) = \text{NF}\mathcal{T}(M_i) \quad (9)$$

where we finally used the hypothesis. From (8), (9) and Lemma A.17 we finally conclude that $t \in \text{nf}_r(c^\bullet(\vec{w}^1, \dots, \vec{w}^n)) \subseteq \text{NF}\mathcal{T}(C(M_1, \dots, M_n))$. \square

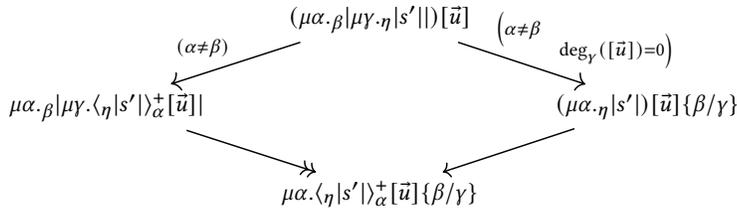


Figure 3. Notable diagrams of Proposition 2.26, point (1), subcase $t = (\mu\alpha.\beta|s|)|[\vec{u}]$, $\mathbb{T} = \mu\alpha.\langle\beta|s| \rangle_{\alpha}^+)|[\vec{u}]$, $s = \mu\gamma.\eta|s'|$, $\alpha \neq \beta$.

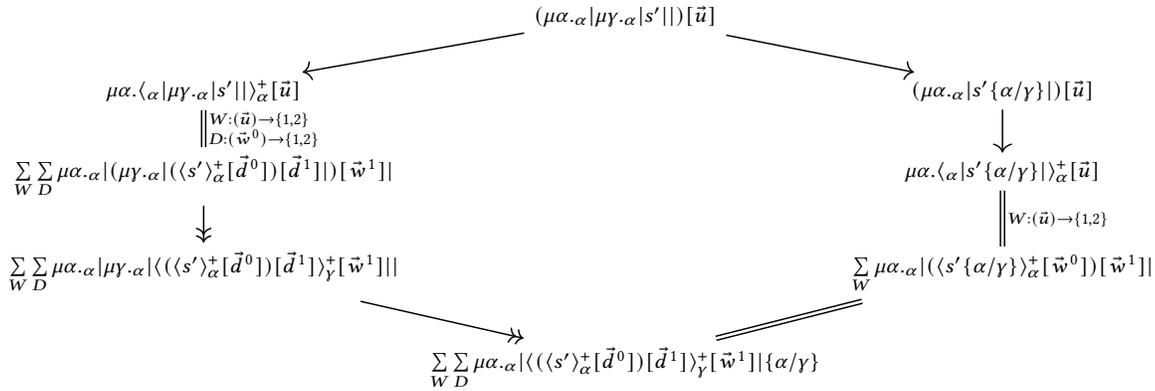


Figure 4. Notable diagrams of Proposition 2.26, point (1), subcase $t = (\mu\alpha.\beta|s|)|[\vec{u}]$, $\mathbb{T} = \mu\alpha.\langle\beta|s| \rangle_{\alpha}^+)|[\vec{u}]$, $s = \mu\gamma.\eta|s'|$, $\alpha = \beta$, $\gamma \neq \eta$, $\eta = \alpha$.

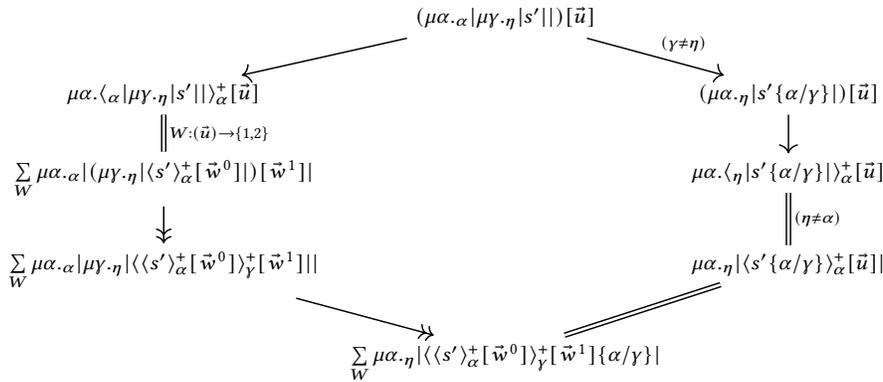


Figure 5. Notable diagrams of Proposition 2.26, point (1), subcase $t = (\mu\alpha.\beta|s|)|[\vec{u}]$, $\mathbb{T} = \mu\alpha.\langle\beta|s| \rangle_{\alpha}^+)|[\vec{u}]$, $s = \mu\gamma.\eta|s'|$, $\alpha = \beta$, $\gamma \neq \eta$, $\eta \neq \alpha$.

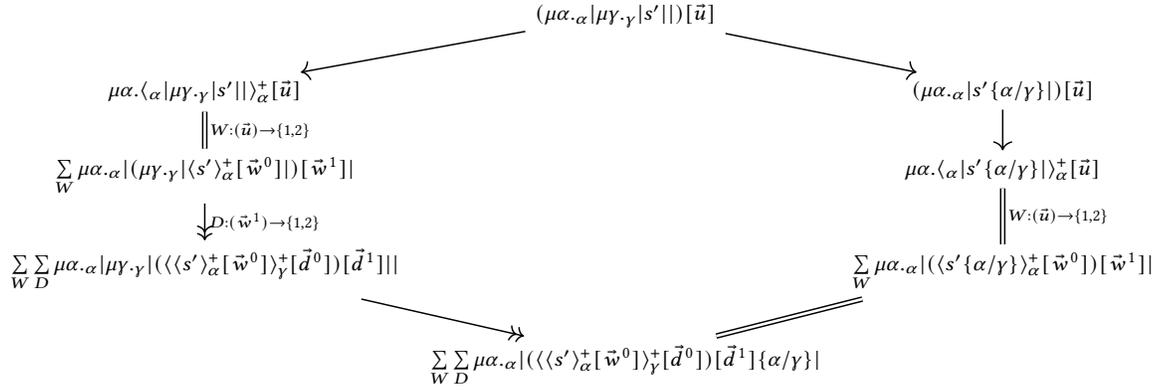


Figure 6. Notable diagrams of Proposition 2.26, point (1), subcase $t = (\mu\alpha.\beta|s|)|[\bar{u}]$, $\mathbb{T} = \mu\alpha.\langle\beta|s| \rangle_{\alpha}^{+}[\bar{u}]$, $s = \mu\gamma.\eta|s'|$, $\alpha = \beta$, $\eta = \gamma$.

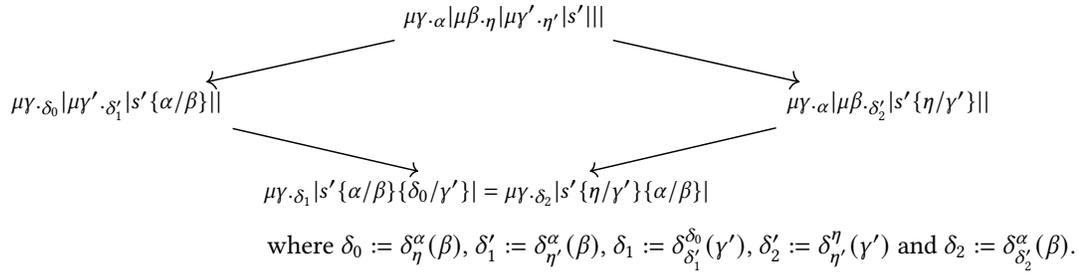


Figure 7. Notable diagrams of Proposition 2.26, point (1), subcase $t = \mu\gamma.\alpha|\mu\beta.\eta|s|$, $\mathbb{T} = \mu\gamma.\eta|s|\{\alpha/\beta\}$, $s = \mu\gamma'.\eta'|s'|$.