Lecture 1
From Computability to Program Theory, Part 1
Topology/domain-theory of the "graph model" over $\mathcal{P}(\mathbb{N})$

Davide Barbarossa

Contents

1 Scott-topology

Let $\uparrow : R \to \mathcal{P}(R)$ defined by $\uparrow a := \{b \in R \mid b \supseteq a\}$ (the principal filter at a).

Ex. 1 — Show that we have:

(i) $R = \uparrow \emptyset$

(ii) $a \in \uparrow a$ for all $a \in R$

(iii) $\uparrow$ is non-increasing wrt. inclusion

(iv) $\uparrow(a \cup b) = \uparrow a \cap \uparrow b$.

Answer (Ex. 1) — See Section 4. □

Let $\mathcal{B} := \mathrm{Im}(\uparrow|_{R_{fin}}) = \{\uparrow e \subseteq R \mid e \subseteq_{fin} \mathbb{N}\} \subseteq \mathcal{P}(R)$.

In other words, each $E \in \mathcal{B}$ is uniquely determined by some finite $e \subseteq_{fin} \mathbb{N}$, in the sense that for all $b \in R$, one has $b \in E$ *iff* $b \supseteq e$.

Proposition 1.1. *$\mathcal{B}$ is a base of a topology on R.*

Proof. By standard undergraduate topology, we have to show that $(i) : \mathcal{B}$ covers R and $(ii) :$ for all $A, B \in \mathcal{B}$ and $c \in A \cap B$, there is $C \in \mathcal{B}$ such that $c \in C \subseteq A \cap B$. This is left as:

Ex. 2 — Complete the proof of Proposition 1.1.

Answer (Ex. 2) — See Section 4. □

Remark 1.2. *By definition, the open sets of the topology induced by $\mathcal{B}$ are exactly the sets $O \subseteq R$ of shape $O = \bigcup_{e \in I} \uparrow e$, for some $I \subseteq R_{fin}$. In particular, if $e \in R_{fin}$, then $\uparrow e$ is open in R.*

This topology is sometimes called *positive information topology* (cfr Topological properties of concept spaces). Intuitively, one thinks of the elements of R as pieces of information, call them concepts, and a basic open ($\uparrow e$) is the collection of concepts that we can access as soon as we can access some finite concept (e).

Proposition 1.3. *The open sets of the topology induced by $\mathcal{B}$ are exactly the $O \subseteq R$ which satisfy the following condition for all $a \in R$:*

$$a \in O \Leftrightarrow O \ni e \subseteq_{fin} a, \text{ for some } e \in R. \tag{1}$$

Proof. We have to show that: there is $I \subseteq R_{fin}$ such that $O = \bigcup_{e \in I} \uparrow e$ iff O satisfies Property (1) for all $a \in R$. Let us show the two implications of the iff.

($\Rightarrow$): Let $I \subseteq R_{fin}$, $O = \bigcup_{e \in I} \uparrow e$ and $a \in R$. We show the two implications of (1).

If $a \in O$ then there is $e \in I$ such that $a \supseteq e$. In particular, e is finite. Also, $e \in \uparrow e$, so $e \in O$.

If $O \ni e \subseteq_{fin} a$ for some $e \in R$, then by definition of O there is $e' \in I$ such that $e \supseteq e'$. By Exercise 1(ii,iii) $\uparrow e' \supseteq \uparrow e \supseteq \uparrow a \ni a$. Then by definition of O we have $a \in O$.

($\Leftarrow$): Assuming (1) for all $a \in R$, we have to show that O can be written as $O = \bigcup_{e \in I} \uparrow e$, for some $I \subseteq R_{fin}$. Consider the function $I : O \to \mathcal{P}(R_{fin})$ defined by $I(a) := \{e \in R_{fin} \mid a \supseteq e \in O\}$. Let $I := \bigcup_{a \in O} I(a) \subseteq R_{fin}$. We claim that $O = \bigcup_{e \in I} \uparrow e$. Let us show the two inclusions:

If $a \in O$ then by (1)($\Rightarrow$) there is $e \in I(a)$. So $e \in I$ and $a \in \uparrow e$ and we are done.

If $a \in \bigcup_{e \in I} \uparrow e$, then there is $e \in R_{fin}$ such that $a \supseteq e$, with $e \in I(b)$ for some $b \in O$. In particular $e \in O$, and by (1)($\Leftarrow$) we have $a \in O$. □

Notice that, from the above, it immediately follows that an open O is $\subseteq$-upward-closed.

Remark 1.4. *We immediately see that R_{fin} is dense in R since, from the proposition above, every non-empty open set of R has non-empty intersection with R_{fin}.*

Remark 1.5. *It is also easy to see that the open neighbourhoods of a point $a \in R$ are exactly those sets of shape $\uparrow e \cup \bigcup_{d \in I} \uparrow d$, for $e \subseteq_{\mathrm{fin}} a$ and $I \subseteq R_{fin}$. From this, it follows that the neighbourhoods of a are exactly the sets $\uparrow e \cup A$, for $e \subseteq_{\mathrm{fin}} a$ and $A \subseteq R$. In order to know if a given $b \in R$ belongs to a fixed neighbourhood of $a \in R$ then, it is enough to test if it contains a fixed finite number of elements of a.*

Proposition 1.6. *R is T_0 (i.e. any two different points are distinguished by the topology, i.e. have different neighbourhoods).*

Proof. Let $a \neq b$ in R. We have to show that there is an open set which contains exactly one among a and b. Since $a \neq b$, there is $n \in a - b$ (or in $b - a$, which is the same). Now $\uparrow\{n\}$ is open by Remark 1.2. Moreover, $\uparrow\{n\} \ni a$ since $a \supseteq \{n\}$, and $\uparrow\{n\} \not\ni b$ since $b \not\supseteq \{n\}$. □

Specialization orders

Definition 1.7. *The* specialization preorder *of a topology is defined by setting $a \leq b$ iff $N(a) \subseteq N(b)$, where $N(c)$ is a local base at c. That is, $a \leq b$ iff $a \in O \Rightarrow b \in O$ for all open O.*

Remark that open sets of a topological space X are upward-closed wrt $\leq$, by definition of $\leq$.

The specialization preorder is always a preorder (i.e. reflexive and transitive, which are immediate), but in general antisymmetry fails, so we do not have a poset. However, we have:

Ex. 3 — A space is T_0 iff it is a poset with the specialization order. Hence, $(R, \leq)$ is a poset.

Answer (Ex. 3) — See Section 4. □

Lemma 1.8. *The specialization order of R coincides with set-theoretic inclusion. That is, for $a, b \in R$, we have $a \subseteq b$ iff $a \leq b$.*

Therefore, we immediately have that every non-empty subset D of R admits sup $\bigcup D$ in R.

Proof. $(\Rightarrow)$: Suppose $a \subseteq b$ and let O open with $a \in O$. We have already remarked that O is $\subseteq$-upward closed, so we are done.

$(\Leftarrow)$: Let us start by assuming that $a \in R_{fin}$. Then $\uparrow a$ is open by Remark 1.2 and contains a by Exercise 1. So from $a \leq b$ we have $b \in \uparrow a$, i.e. $b \supseteq a$. Now for the case of a generic $a \in R$ (not necessarily finite), let $n \in a$. Then $\{n\} \subseteq_{fin} a$, so by the $(\Rightarrow)$ just proved, $\{n\} \leq a$. But $a \leq b$ by hypothesis, $\{n\} \leq b$. Now by the finite case we just proved, $\{n\} \subseteq b$, i.e. $n \in b$. □

Remark 1.9. *A space is T_1 if $x \neq y \Rightarrow \exists$ open U, V s.t. $x \in U \not\ni y$ and $x \notin V \ni y$. Clearly, Hausdorff $\Rightarrow T_1 \Rightarrow T_0$. Also, in a T_1 space, $x \leq y \Rightarrow x = y$. Hence, R is not T_1 nor Hausdorff.*

Remark 1.10. *We have seen that a topology τ always induces a preorder $\leq_\tau$, namely its specialization preorder.*

Conversely, given a partial order $\preceq$, one can always take the finest topology which has $\preceq$ as specialization preorder, called the Alexandroff topology of $\preceq$, *and the coarser topology with the same property, sometimes called the* upper topology of $\preceq$.

In our case R, we have seen that the specialization partial order of our topology is $\subseteq$. However, our topology is neither *the Alexandroff topology of the inclusion,* not *the upper topology of the inclusion. In fact, there is another intermediate solution to the problem of defining a topology that has specialisation poset a given poset, and it is consists in taking its* Scott topology. *In the next paragraph, we are going to see that our topology is indeed the Scott topology of the inclusion.*

Scott-topology

Definition 1.11. *A subset D of a poset X is called* directed *whenever $D \neq \emptyset$ and for all $d, d' \in D$, $\{d, d'\}$ admits an upper bound in D.*

Example 1.12. *Let $a \in R$. Then $\bigcup \downarrow_{fin} a = a$. Moreover, if $a \neq \emptyset$, then $\downarrow_{fin} a$ is $\subseteq$-directed.*

Definition 1.13. *Let $(X, \preceq)$ be a poset. Its* Scott-topology *is defined by declaring open the subsets U of X such that:*

1. *U is upward closed (i.e. $U \ni x \preceq y \Rightarrow U \ni y$)*
2. *for all directed $D \subseteq X$ which admits $\bigvee D \in U$, we have $D \cap U \neq \emptyset$.*

Ex. 4 — Show that the Scott-open sets on a poset $(X, \preceq)$ form a topology on X indeed.

Answer (Ex. 4) — See Section 4. □

* **Ex. 5** — For a poset X, the set $O_h := \{x \in X \mid x \not\preceq h\}$ is open in the Scott-topology.

Answer (Ex. 5) — See Section 4. □

Ex. 6 — The specialisation preorder of a Scott-topology on a poset $(X, \preceq)$ is $(X, \preceq)$.

Answer (Ex. 6) — See Section 4. □

Definition 1.14. *In a poset $(X, \preceq)$, we say that an element $e \in X$ is* compact *iff for all directed $D \subseteq X$ admitting $\bigvee D$, we have: if $e \preceq \bigvee D$ then $e \preceq d$ for some $d \in D$.*

Proposition 1.15. *In $(R, \subseteq)$, the set of compact elements is R_{fin}.*

Proof. Let $e \in R$.

(e compact $\Rightarrow$ e finite): Wlog $e \neq \emptyset$. Then by Example 1.12 $\downarrow_{fin} e$ is directed and admits sup e. But then from $e \subseteq e$, the compactness of e yields $e \subseteq d$ for some $d \in \downarrow_{fin} e$. But the latter says in particular that d is finite, thus e is too (actually, $e = d$).

(e finite $\Rightarrow$ e compact): Let $D \subseteq R$ directed (always admitting $\bigvee D = \bigcup D$, thanks to Lemma 1.8), and suppose $e \subseteq \bigcup D$. We have to show that $e \subseteq d$ for some $d \in D$. Wlog $e \neq \emptyset$, otherwise it is trivial (as $D \neq \emptyset$). Now, there is a function[1] $d_{(\_)} : e \to D$ such that $n \in d_n$ for all $n \in e$. But since e is finite and non-empty, the image $\{d_n \mid n \in e\}$ is a finite non-empty subset of D. But D is directed, so it contains an upper bound of all its finite non-empty subsets, therefore there is $u \in D$ such that $d_n \subseteq u$ for all $n \in e$. We claim that this is our desired set, i.e. we have to show that $e \subseteq u$. Let $n \in e$. So $n \in d_n$, i.e. $\{n\} \subseteq d_n$, so $\{n\} \subseteq u$, i.e. $n \in u$. □

Theorem 1.16. *The topology that we are considering on R coincides with the Scott-topology of the partial order $\subseteq$ in R.*

Ex. 7 — Prove it.

Answer (Ex. 7) — See Section 4. □

Observe that from Theorem 1.16 and Exercise 6 we obtain again Lemma 1.8.

In fact, all the properties that we will mention here do not really depend on the particular space R: they only depend on the fact that R is the Scott topology of a (directed)-cpo (and even algebraic). Since this coincides with its specialisation preorder, one often even drops the topological language and just talks in terms of order theory (which, for those particular orders, is called domain theory). However, we will not develop domain theory except some basic results, and we will mostly concentrate on R by employing a topological language.

2 Scott-Continuous functions

Lemma 2.1. *Let X, Y be topological spaces. If a function $f : X \to Y$ is continuous, then it is monotone with respect to their specialization preorders.*

Ex. 8 — Prove Lemma 2.1

Answer (Ex. 8) — See Section 4. □

Lemma 2.2. *Let X, Y be posets and $f : X \to Y$ monotone. Then f sends directed sets to directed sets.*

[1] Since e is finite we do not even need the axiom of choice for this.

Proof. We have to show that, if D is directed in X, then fD is directed in Y. That $fD \neq \emptyset$ is trivial from the fact that, being D directed, $D \neq \emptyset$. Now let $x, x' \in D$. Since D is directed there is $c \in D$ upper bound of x, x'. Thus, $f(c)$ is a desired upper bound of $f(x), f(x')$ in fD. □

Theorem 2.3. *Let X, Y be posets and $f : X \to Y$. The following are equivalent:*

i): f is continuous wrt the Scott-topologies on X, Y.

ii): For all directed $D \subseteq X$ admitting sup $\bigvee D \in X$, there exists $\bigvee(fD)$ in Y and it is:

$$\bigvee(fD) = f(\bigvee D). \qquad \text{(Scott-Continuity)}$$

Proof. $((i) \Rightarrow (ii))$: In order to show that $f(\bigvee D) \in Y$ is the sup of fD in Y we need to show that, first, $f(\bigvee D) \leq f(d)$ for all $d \in D$. This is immediate because f is continuous, so by Lemma 2.1 it is monotone wrt its specialisation preorder, and we know from Exercise 6 that such preorder is the one on X that we started with. Second, we have to show that $f(\bigvee D)$ is the minimum of all upper bounds y of fD. Given such y, we need to show that $f(\bigvee D) \leq y$. But this is the same as showing $f(\bigvee D) \notin O_y$, where the latter is the open in Y defined in Exercise 5. Now for showing the desired result, suppose $f(\bigvee D) \in O_{y)}$. Then $\bigvee D \in f^{-1}O_y$. But the latter is open in X, since O_y is open in Y and f is continuous. Therefore, by Definition 1.13(2), there is $d \in D$ that belongs to $f^{-1}O_y$, i.e. $f(d) \not\leq y$. But this is impossible, because y is an upper bound of fD.

$((ii) \Rightarrow (i))$: Let us first show that f is $\preceq$-monotone. For this, let $x \preceq x'$. It is trivial that $\downarrow x'$ is directed in X and that there is $\bigvee \downarrow x' = x' \in X$. By Scott-continuity, there is $\bigvee(f \downarrow x') = f(\bigvee \downarrow x') = f(x')$. But since $x \preceq x'$, we have $f(x) \in f \downarrow x'$, and therefore $f(x) \preceq \bigvee(f \downarrow x') = f(x')$, and we are done. Now let us show the continuity of f. For this, let U be Scott-open in Y. In order to show that $f^{-1}U$ is open in X, let us first show that the latter is $\preceq$-upper closed. For this, let $x \preceq x'$ in X such that $f(x) \in U$. By monotonicity, $U \ni f(x) \preceq f(x')$, and because U is Scott-open, we have $f(x') \in U$. Second, we have to show that, given D directed in X admitting sup $\bigvee D \in f^{-1}U$, we have $D \cap f^{-1}U \neq \emptyset$. By Scott-continuity there is $\bigvee(fD) = f(\bigvee D) \in U$. Now by Lemma 2.2, fD is directed in Y. But since U is Scott-open in Y and $\bigvee(fD) \in U$ for fD directed in Y, there is $b \in fD \cap U$. This means that there is $d \in D$ such that $f(d) \in U$, i.e. $d \in f^{-1}U$, and we are done. □

Theorem 2.4. *Let $f : R \to R$. The following are equivalent:*

(i) f is continuous.

(ii) f is monotone for $\subseteq$ and, for all $a \in R$ and $d \in R_{fin}$, the following property (S) holds: if

$$d \subseteq f(a)$$

then there is $e \in R_{fin}$ such that

$$e \subseteq a \qquad \textit{and} \qquad d \subseteq f(e).$$

(iii) for all $a \in R$ and $d \in R_{fin}$ we have:

$$d \subseteq f(a)$$

iff there is $e \in R_{fin}$ such that

$$e \subseteq a \qquad \textit{and} \qquad d \subseteq f(e).$$

(iv) f *is Scott-continuous (i.e. preserves direct unions) from* $(R, \subseteq)$ *to* $(R, \subseteq)$*.*

Proof. From Theorem 2.3 and Theorem 1.16 we have $(i) \Leftrightarrow (iv)$. Let us do the other equivalences.

$(i) \Rightarrow (ii)$: The fact that f is non-decreasing is Lemma 2.1 and Lemma 1.8. Let now $a \in R$ and $d \in R_{fin}$. If $d \subseteq f(a)$, then $a \in f^{-1} \uparrow d$; but $\uparrow d$ is open by Remark 1.2, so by continuity also $f^{-1} \uparrow d$ is open. Then by Proposition 1.3 there is $e \in R_{fin}$ with $f^{-1} \uparrow d \ni e \subseteq a$. In particular, $f(e) \supseteq d$, and we are done.

$(ii) \Rightarrow (iii)$: Let $a \in R$ and $d \in R_{fin}$. The left to right direction of the "iff" is exactly as in the previous case (in particular, we do not use monotonicity of f). Let us now see the other implication. Let $e \in R_{fin}$ with $e \subseteq a$ and $d \subseteq f(e)$. By monotonicity $f(e) \subseteq f(a)$, so $f(a) \supseteq d$ and we are done (in particular, we did not use the hypothesis (S)).

$(iii) \Rightarrow (i)$: Let O open, and we have to show that $f^{-1}O$ is open i.e., by Proposition 1.3, that given $a \in R$, we have: $f(a) \in O$ iff there is $e \in R_{fin}$ with $e \subseteq a$ and $f(e) \in O$.

From left to right: if $f(a) \in O$, by Proposition 1.3 there is $d \in R_{fin}$ such that $O \ni d \subseteq f(a)$. By (iii) there is $e \in R_{fin}$ such that $e \subseteq a$ and $d \subseteq f(e)$. In particular, since O is upward-closed because it is open, $f(e) \in O$.

From right to left: let $e \in R_{fin}$ with $e \subseteq a$ and $f(e) \in O$. By Proposition 1.3 there is $d \in R_{fin}$ such that $O \ni d \subseteq f(e)$. By (iii) we have $d \subseteq f(a)$. In particular, since O is upward-closed because it is open, $f(a) \in O$. ☐

Remark 2.5. *As a consequence of the theorem above, from the Scott-continuity and the fact that* $\bigcup_{e \subseteq_{fin} a} e = a$ *of Example 1.12, all continuous* $f : R \to R$ *satisfies:*

$$f(a) = \bigcup_{e \subseteq_{fin} a} f(e).$$

We thus precisely have the approximation notion that we were looking for in the introduction. In particular, remark that now a function $R_{fin} \to R$ does *(uniquely) define a function* $[R \to R]$*, because the union (the limit, in this topology) is always defined.*

As one could guess, a similar characterisation can be given for Scott-continuous functions on arbitrary (directed-complete-)posets, instead of R, and with compact elements instead of R_{fin}.

3 Encoding (continuous) functions as points

Now that we found our notion of "nice" functions, with their correct notion of approximation, we can proceed with the injection of the set of those nice functions into the base space.

Definition 3.1. *Define the following encodings, clearly computable:*

- $\mathrm{pair} : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, $\mathrm{pair}(n, m) := 2^n(2m + 1) - 1$.
- $\mathrm{list} : \mathbb{N}^* \to \mathbb{N}$, $\mathrm{list}([]) := 0$ *and* $\mathrm{list}(n :: l) := 1 + \mathrm{pair}(n, \mathrm{list}(l))$.

So we have a computable encoding $\langle \_, \_ \rangle : \mathbb{N}^* \times \mathbb{N} \to \mathbb{N}$*, defined by* $\langle l, n \rangle := \mathrm{pair}(\mathrm{list}(l), n)$*.*

* **Ex. 9** — Write their computable decoding functions unpair : $\mathbb{N} \to \mathbb{N} \times \mathbb{N}$, unlist : $\mathbb{N} \to \mathbb{N}^*$ showing that they are bijective. Conclude that also $\langle \_, \_ \rangle$ is a computable and bijective decoding.

Answer (Ex. 9) — See Section 4. ☐

Definition 3.2. *Define the following maps:*

- $\mathrm{set} : \mathbb{N}^* \to R_{fin}$*,* $\mathrm{set}(n_1, \ldots, n_k) := \{n_1, \ldots, n_k\}$
- $\mathrm{kl} : R \to \mathcal{P}(\mathbb{N}^*)$*,* $\mathrm{kl}(a) := \{l \in \mathbb{N}^* \mid \mathrm{set}(l) \subseteq a\}$*. This map sends RE sets to RE sets.*

Definition 3.3. *Define the following functions:*

- $\mathit{fun} : R \to (R \Rightarrow R)$*,*

  $$\mathit{fun}(a)(b) := \{n \in \mathbb{N} \mid \text{ there is } l \in \mathrm{kl}(b) \text{ such that } \langle l, n\rangle \in a\}.$$

  We also define

  $$@ := \mathrm{uncurry}(\mathit{fun}) : R \times R \to R.$$

- $\widetilde{\lambda} : R^R \to R$*,*

  $$\widetilde{\lambda}(f) := \{\langle l, m\rangle \in \mathbb{N} \mid l \in \mathbb{N}^*,\, m \in f(\mathrm{set}(l))\}.$$

  We then define

  $$\lambda := \widetilde{\lambda}\big|_{R \Rightarrow R} : (R \Rightarrow R) \to R$$

  where $R \Rightarrow R$ *is the set of continuous functions from* R *to itself.* $\lambda(f)$ *is sometimes called the* trace *of* f*, as it provides the same exact information of* f*, but encoded in a finitary way inside* R*.*

Proposition 3.4. *We indeed have* $\mathrm{Im}(\mathit{fun}) \subseteq (R \Rightarrow R)$*, as claimed in the definition.*

Proof. Let $a \in R$. In order to show that $\mathit{fun}(a) : R \to R$ is continuous, we use Theorem 2.4 and we only need to show that, given $b \in R, d \in R_{fin}$, we have $d \subseteq \mathit{fun}(a)(b)$ iff $d \subseteq \mathit{fun}(a)(e)$ for some $e \subseteq_{fin} b$.

($\Leftarrow$) : Let $e \subseteq_{fin} b$ and $n \in d \subseteq \mathit{fun}(a)(e)$. By definition of *fun* there is $l_n \in \mathrm{kl}(e)$ (i.e. $\mathrm{set}(l_n) \subseteq e$) with $\langle l_n, n\rangle \in a$. But it is immediate that $\mathrm{kl}(e) \subseteq \mathrm{kl}(b)$, so $n \in \mathit{fun}(a)(b)$.

($\Rightarrow$) : Suppose $d \subseteq \mathit{fun}(a)(b)$. Then, for all $n \in d$ there is, by definition of *fun*, a list $l_n \in \mathrm{kl}(b)$ with $\langle l_n, n\rangle \in a$. So we have a function $l_{(\_)} : d \to \mathrm{kl}(b)$. But since both $\mathrm{set}(l_n)$ and d are finite, we have $e := \bigcup_{n \in d} \mathrm{set}(l_n) \subseteq_{fin} b$. Now let $n \in d$. In order to show that $n \in \mathit{fun}(a)(e)$, we can find $l' \in \mathrm{kl}(e)$ with $\langle l', n\rangle \in a$. This is trivial by taking $l' := l_n$. □

Proposition 3.5. $@ : R \times R \to R$ *is continuous (wrt the product topology).*

Proof. Remember that the product topology has $\{O_1 \times O_2 \mid O_1, O_2 \text{ open in } R\}$ as a basis. Let O open in R. We show that $@^{-1}O = \{(a, b) \in R \times R \mid \mathit{fun}(a)(b) \in O\} = \bigcup_{a \in R}(\{a\} \times \mathit{fun}(a)^{-1}O)$ is open. We claim that $\bigcup_{a \in R}(\{a\} \times \mathit{fun}(a)^{-1}O) = \bigcup_{e \in R_{fin}}(\uparrow e \times \mathit{fun}(e)^{-1}O)$. Once this is proven, we are done, because $\uparrow e$ is open by Remark 1.2 and $\mathit{fun}(e)^{-1}O$ is open because $\mathit{fun}(e)$ is continuous, so $@^{-1}O$ would be a union of elements of the canonical base of the product topology. We leave the two inclusions as the following:

* **Ex. 10** — Conclude the proof of Proposition 3.5.

Answer (Ex. 10) — See Section 4. □

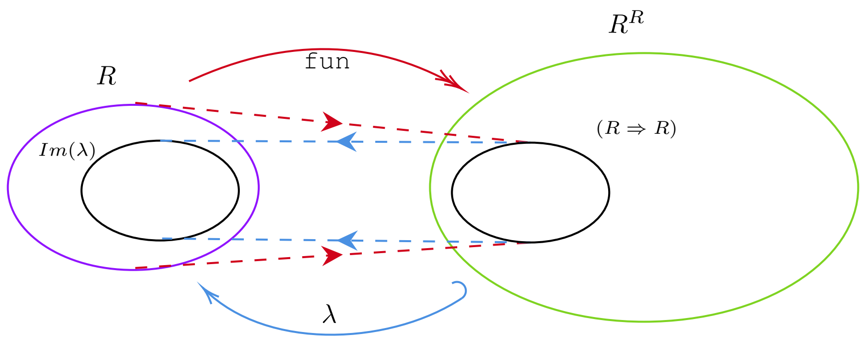

Figure 1: Retraction of R onto $\mathrm{Im}(\lambda) \approx (R \Rightarrow R)$ via $\lambda \circ \mathit{fun}$.

Theorem 3.6. *We have:*

$$\mathit{fun} \circ \lambda = \mathrm{id}_{R \Rightarrow R}. \tag{β}$$

Moreover,

$$\lambda \circ \mathit{fun} \supseteq \mathrm{id}_R \tag{$\widetilde{\eta}$}$$

Finally, for all $f \in (R \Rightarrow R)$, we have:

$$\lambda(f) = \bigcup_{\mathit{fun}(a) = f} a.$$

Proof. The "finally" claim is immediate: the $(\subseteq)$ is immediate since, by (β), $\lambda(f)$ is precisely one of those a. For $(\supseteq)$, if $\mathit{fun}(a) = f$, then using $(\widetilde{\eta})$ we have $\lambda(f) = \lambda(\mathit{fun}(a)) \supseteq a$.

Let us prove the two equations.

(β) : We have to prove that $\mathit{fun}(\lambda(f))(b) = f(b)$ for all $b \in R$. Let us first remark that, by definition of fun, λ, given $n \in \mathbb{N}, b \in R$ we have: $n \in \mathit{fun}(\lambda(f))(b)$ iff there is $l \in \mathbb{N}^*$ such that $\mathrm{set}(l) \subseteq b$ and $n \in f(\mathrm{set}(l))$. Now we can do the two inclusions:

For $(\subseteq)$, let $n \in \mathit{fun}(\lambda(f))(b)$, i.e. (by what we just remarked) $n \in f(\mathrm{set}(l))$ for some $l \in \mathbb{N}^*$ with $\mathrm{set}(l) \subseteq b$. But f is $\subseteq$-monotone by Theorem 2.4, so $f(\mathrm{set}(l)) \subseteq f(b)$. Thus $n \in f(b)$ and we are done. For $(\supseteq)$, let $n \in f(b)$, i.e. $\{n\} \subseteq_{\mathrm{fin}} f(b)$. Since f is continuous, by Theorem 2.4 there is $e \subseteq_{fin} b$ such that $n \in f(e)$. But since e is finite, we can take any enumeration of e and we have a list $l_e \in \mathbb{N}^*$ such that $\mathrm{set}(l_e) = e \subseteq b$ and $n \in f(e) = f(\mathrm{set}(l_e))$. By what we remarked above, this means that $n \in \mathit{fun}(\lambda(f))(b)$ and we are done.

$(\widetilde{\eta})$: Let us remark that, given $n \in R$, we have: $n \in \lambda(\mathit{fun}(a))$ iff there is $l \in \mathbb{N}^*, m \in \mathbb{N}, h \in \mathrm{kl}(\mathrm{set}(l))$ such that $n = \langle l, m \rangle$ and $\langle l, m \rangle \in a$ Now given $n \in a$, we trivially obtain the result by taking $l = h := \mathrm{unpair}(\mathrm{unpair}(n)_1)$ and $m := \mathrm{unpair}(n)_2$. □

If we had an equality in equation $(\widetilde{\eta})$, we would call it the η-equation. So R does *not* satisfy (η).

Definition 3.7. *The structure $(R, \lambda, @)$ is called the*[2] graph model.

In the next lecture we are going to see what and in which sense the graph model is a model of.

[2] Actually, there are many different graph models, and this is only one of them. They all have a similar construction, that can be carried out for other sets than $R = \mathcal{P}(\mathbb{N})$. Even for $\mathbb{N}$, the particular encodings that we choose are just one choice among many possible. However, some properties of the graph model depend on the choice of the encodings, and the ones that we have taken here are among the "good" choices.

Remark 3.8. *Remark that from (β) it follows that λ is injective and fun is surjective. That is, as shown in Figure 1, that we have injected a space of functions into its base space. However, we would like fun and λ be continuous, otherwise the so-called retraction of R onto R ⇒ R is not necessarily interesting – say one defines fun arbitrarily out of the image of* $(R \Rightarrow R)$*. In order to talk about continuity of them, one has to discuss the topology on the function space* $R \Rightarrow R$*. This can done by seeing R as an object of the category of algebraic cpo's and* $R \Rightarrow R$ *is an exponential object in that category. However, we will not do it now, and we take it as granted: then we have that the pair* (λ, fun) *defines a* topological retraction *of R onto* $\mathrm{Im}(\lambda) \approx [R \to R]$.

Remark 3.9. *Remark that we have retracted a space of functions onto its base space (intuitively speaking, the topology identifies a functional space with 1 dimension). We could also retract* $R \times R$ *onto R (intuitively speaking, the topology identifies 2 dimensions with 1 dimension). Therefore, the topology we are considering doesn't really carry a geometrical meaning, but rather it has to be understood as a handy way for talking about order theoretic notions (in fact, domain theory).*

Proposition 3.10. *All continuous* $f : R \to R$ *admit fixed points. More specifically, the function, which is called a* fixed point combinator*,*

$$Y : (R \Rightarrow R) \to R \qquad Y(f) := @(\delta(\Delta_f))$$

where $\delta : R \to R \times R$ *is the diagonal* $\delta(a) := (a, a)$ *and* $\Delta_f := \lambda(f \circ @ \circ \delta) \in R$*, gives one fixed point of its input function, i.e. for all f we have:*

$$f(Y(f)) = Y(f).$$

Proof. Immediate by using β-equation: $Y(f) = @(\delta(\Delta_f)) = \mathit{fun}(\Delta_f)(\Delta_f) = \mathit{fun}(\lambda(f \circ @ \circ \delta))(\Delta_f) = (f \circ @ \circ \delta)(\Delta_f) = f(@(\delta(\Delta_f))) = f(Y(f))$. □

Theorem 3.11. *The set of RE sets is closed wrt the following rules:*

$$\frac{}{\lambda(\lambda \circ \mathrm{curry}(\overset{(n-1)}{\cdots}(\lambda \circ \mathrm{curry}(\mathrm{proj}_i^n))\cdots))} \qquad \frac{f \in (R \Rightarrow R) \text{ and computable}}{\lambda(f)} \qquad \frac{a \quad b}{@(a, b)}$$

Comments on the proof. Try to convince yourself that any set defined by the rules above is RE. This is not trivial but relatively straightforward using Scott-continuity and the fact that the encodings previously defined preserve the fact of being RE. □

In the next lecture we will take inspiration from the previous theorem in order to introduce the λ-calculus, by mimicking the three cases above: In fact, the previous theorem tells us *how* how we can inductively define a subset of all RE sets starting from basic pieces via some constructions. In particular, the two inductive rules handle high-order encoding of functions. Therefore, it makes sense to take precisely those basic pieces and constructions as a way of putting together instructions for RE sets, i.e. how to put together pieces of code, i.e. how to build *programs*! We will therefore take them as formal objects and declare them to be a programming language.

4 Solutions to the exercises

Answer (Ex. 1) — All are immediate, since for all $a, b, c \in R$ we have: (i): $a \supseteq \emptyset$, (ii): $a \supseteq a$; (iii): $a \subseteq b \Rightarrow \uparrow a \supseteq \uparrow b$; (iv): $c \in \uparrow(a \cup b)$ iff both $c \supseteq a$ and $c \supseteq b$ iff $c \in \uparrow a \cap \uparrow b$. □

Answer (Ex. 2) — (i): We have to show that $R = \bigcup_{e \in R_{fin}} \uparrow e$. ($\subseteq$): Exercise 1(i). ($\supseteq$): there is nothing to prove.
(ii): Let $A = \uparrow a$ and $B = \uparrow b$, for $a, b \in R_{fin}$. Then $a \cup b \in R_{fin}$ and thus $C := \uparrow(a \cup b) \in \mathcal{B}$. By Exercise 1(ii) $C = A \cap B$ and so we are done. □

Answer (Ex. 3) — We have $x \leq y \leq x$ in the specialization preorder iff x, y have exactly the same open neighborhoods. So the antisymmetry of $\leq$ is exactly the contrapositive of the T_0 property. □

Answer (Ex. 4) — All is immediately checked. The only non-immediate check is the closure of the second property of Scott-opens wrt finite intersections: Let I be a *finite* collection of Scott-open sets in X, and let D be directed in X admitting $\bigvee D \in \bigcap_{U \in I} U$. We need to show that $D \cap \bigcap_{U \in I} U \neq \emptyset$. For all $U \in I$, we have $\bigvee D \in U$, so since U is Scott-open there is $d_U \in D \cap U$. This defines a function[3] $d_{(\_)} : I \to D$. Since I is finite and D is directed, there is $v \in D$ which upper bounds all d_U for $U \in I$. Moreover, for all $U \in I$, we have $U \ni d_U \leq v$ and, since U is $\leq$-upward closed because it is Scott-open, we obtain $v \in U$. Therefore, $v \in \bigcap_{U \in I} U$. □

Answer (Ex. 5) — 1) of Definition 1.13: let $x \leq x'$ and we show that $x \in O_h \Rightarrow x' \in O_h$. We show the contrapositive: if $x' \notin O$, then $x' \leq h$, but then $x \leq x' \leq h$, so $x \notin O_h$.
2) of Definition 1.13: Let B directed in X, and we show that $\bigvee B \in O_h \Rightarrow B \cap O_h \neq \emptyset$. We show the contrapositive: if $B \cap O_h = \emptyset$ then for all $x \in B$, $x \leq h$, so by definition of sup we have $\bigvee B \leq h$, so $\bigvee B \notin O_h$. □

Answer (Ex. 6) — We have to show that $x \leq y$ in the specialisation preorder iff $x \preceq y$ in X. ($\Rightarrow$): We have to show that $x \preceq y$. Consider O_y of the Exercise 5, which is open. Remark also that, in general, $y \notin O_y$. But then since $x \leq y$ in the specialisation preorder, taking the contrapositive of its definition we have that $x \notin O_y$. And this means that $x \preceq y$. ($\Leftarrow$): We have to show that, given O open, if $x \in O$ then $y \in O$. But this is trivially Definition 1.13(1). □

Answer (Ex. 7) — Let $O \subseteq R$. We need to prove that O is Scott-open iff it satisfies (1) for all $a \in R$.
($\Rightarrow$) : Let $a \in R$. The right to left of (1) is immediate because O is $\subseteq$-upward closed by definition of Scott-open. For the left to right of (1), suppose $a \in O$. Now if $O = R$, then it is trivial, because $O = R \ni \emptyset \subseteq_{fin} a$. If $O \neq R$, then $a \neq \emptyset$. Indeed, if $a = \emptyset$ then because O is upward closed, we have $O = R$, which is not the case. But now since $a \neq \emptyset$, Example 1.12 says that $\downarrow_{fin} a$ is directed in R and admits $a \in O$ as sup. Then by definition of Scott-open there is $e \in \downarrow_{fin} a \cap O$, i.e. $O \ni e \subseteq_{fin} a$.
($\Leftarrow$) : We already remarked that the open O is $\subseteq$-upward closed. Let now $D \subseteq R$ be directed with $\bigcup D \in O$, and we have to show that $D \cap O \neq \emptyset$. By (1) there is $e \in O$ such that $e \subseteq_{fin} \bigcup D$. But being finite, e is compact by Proposition 1.15, so there is $d \in D$ such that $e \subseteq d$. But since

[3]Since I is finite we do not even need the axiom of choice for this.

we just saw that O is $\subseteq$-upward closed (or, equivalently, using the other side of (1)), we have $d \in O$ and we are done. $\square$

Answer (Ex. 8) — Let $\leq, \leq'$ the specialization preorders of X and Y, respectively. Let $x \leq y$ in X and O be open in Y. If $f(x) \in O$ then $x \in f^{-1}O$, which is open in X because f is continuous, so $y \in f^{-1}O$ because $x \leq y$, i.e. $f(y) \in O$. We have proved that $f(x) \leq' f(y)$. $\square$

Answer (Ex. 9) — They are given by the following programs.

```
unpair(j : ℕ) : ℕ × ℕ

k ← j + 1
n ← 0
while(k ≥ 1) do
  if(k even)
    then
      k ← k/2
      n ← n + 1
    else
      return  (n , (k−1)/2)
od
```

```
unlist : ℕ → ℕ*

unlist(0)        :=  []
unlist(j + 1)    :=  let (n, m) := unpair(j)
                       in n :: unlist(m)
```

$\square$

Answer (Ex. 10) — $(\subseteq)$: Let $(a, b) \in \{a\} \times \mathit{fun}(a)^{-1}O$, i.e. $\mathit{fun}(a)(b) \in O$. But by Proposition 1.3, there is $h \subseteq_{\mathit{fin}} \mathit{fun}(a)(b)$ with $h \in O$. By definition of fun, this means that we have a function $l_{(\_)} : h \to \mathrm{kl}(b)$ such that $\langle l_n, n\rangle \in a$ for all $n \in h$. Let now $e := \{\langle l_n, n\rangle \mid n \in h\} \in R$. Since h is finite, e is finite. By construction, $e \subseteq a$. Finally, if $n \in h$ then $\langle l_n, n\rangle \in e$ by definition of e, i.e. $h \subseteq \mathit{fun}(e)(b)$ by definition of fun. By Proposition 1.3, since $O \ni h$ finite, $O \ni \mathit{fun}(e)(b)$. In conclusion, we found $e \in R_{\mathit{fin}}$ such that $(a, b) \in \uparrow e \times \mathit{fun}(e)^{-1}O$.
$(\supseteq)$: Let $(a, b) \in \uparrow e \times \mathit{fun}(e)^{-1}O$, with $e \in R_{\mathit{fin}}$, i.e. we have $e \subseteq_{\mathit{fin}} a$ and $\mathit{fun}(e)(b) \in O$. We have to show that $(a, b) \in \{a\} \times \mathit{fun}(a)^{-1}O$, i.e. that $\mathit{fun}(a)(b) \in O$. We remark that, by definition of fun, we have a function $l_{(\_)} : \mathit{fun}(e)(b) \to \mathrm{kl}(b)$ such that $\langle l_n, n\rangle \in e$ for all $n \in \mathit{fun}(e)(b)$. In particular, $e \supseteq \{\langle l_n, n\rangle \mid n \in \mathit{fun}(e)(b)\}$, therefore the latter is finite because e is. But since pair is injective, $\mathit{fun}(e)(b)$ cannot be infinite, otherwise the latter set would be infinite as well, which we just saw is not the case. So we have proven that $\mathit{fun}(e)(b)$ is finite. Moreover, by looking at the definition of fun, we see that $\mathit{fun}(e)(b) \subseteq \mathit{fun}(a)(b)$ because $e \subseteq a$. But then from $O \ni \mathit{fun}(e)(b)$, Proposition 1.3 gives $O \ni \mathit{fun}(a)(b)$. $\square$